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(editors)

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6th CAME Symposium: Improving tools, tasks and teaching in CAS-based mathematics education

THEME 1: IMPROVING TOOLS
Presenter: Gunnar Gjone, *Univerity of Oslo, Norway*
Coordinator and reactor: Djordje Kadijevich

THEME 2: IMPROVING TASKS
Presenter: Margot Berger, *University of Witwatersrand, South Africa*
Coordinator and reactor: Matija Lokar

THEME 3: IMPROVING TEACHING
Presenter: Jean-baptiste Lagrange, *University of Reims, France*
Coordinator and reactor: Robert Mayes

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Preface

Computer Algebra in Mathematics Education (CAME) is an open, international organization founded during a special meeting at the 8th International Congress of Mathematical Education (ICME) in Seville in July 1996. Its main goal is to facilitate the dissemination and exchange of information on research and development in the use of computer algebra in mathematics education. Beginning in 1999, CAME biannually organized symposiums. (See www.lkl.ac.uk/research/came/)

The 6th CAME Symposium: Improving tools, tasks and teaching in CAS-based mathematics education, took place in Belgrade, Serbia, on 16 and 17 July 2009, organized by Megatrend University. The three themes of the Symposium, its committees and sponsors are listed in the Appendix. There were sixteen participants from eleven countries and four continents.

The Proceedings contains papers presented at the Symposium that passed a rigorous refereeing process done by members of the Program Committee, whose useful comments and suggestions were intended to help the authors improve their contributions. We thank these authors for their contributions.

We would like to express our gratitude to Megatrend University for hosting the Symposium and for publishing the Proceedings, especially to Prof. Mića Jovanović, Rector of Megatrend University, and Academician Prof. Gradimir Milovanović, dean of Faculty of Computer Science—the host Faculty of the Symposium as well as to the members of the Organizing Committee. We would also like to express our gratitude to the sponsors, which supported the organization of this Symposium and realized its media presentation.

The Editors

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DESIGNING TASKS FOR CAS CLASSROOMS: CHALLENGES AND OPPORTUNITIES FOR TEACHERS AND RESEARCHERS *

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Abstract. Using a semiotic framework I isolate three key components of a CAS–based mathematical task: construction of signs, experimentation and transformation of signs, and interpretation of the transformed signs. Within each of these components, the design of a CAS–based task has implications for the production or interpretation of mathematical signs. The construction of appropriate CAS–based signs is often problematic for students and the level of guidance in this regard is a challenge for the teacher. At the same time, complexities around the construction of signs may open up new possibilities for task design. Furthermore, certain information in a task statement may profoundly affect students' experimentation and interpretation; the extent to which such information is provided or withheld provides opportunity and challenge for the teacher.

INTRODUCTION

I use a semiotic framework to deconstruct a CAS–based task into its components. I then illustrate how this framework illuminates aspects of the task design, which may promote or hinder mathematical activity. Throughout the talk, I assume that a CAS–based task involves the use of both CAS and paper-and-pencil (hand) work.

THEORETICAL FRAMEWORK

I regard mathematics as a semiotic system, that is, as a system of signs. I assume-Ernest's [1] formulation of- mathematics as consisting of three components: a set of signs which may be written or uttered or encoded electronically, a set of rules for sign production and a "set of relationships between signs and their meanings embodied in an underlying meaning structure" [1, p. 70]. Within a semiotic framework, the idea of the mathematical object and the written, spoken or encoded inscription are mutually constitutive. Neither can exist without the other and both evolve with each other. This view of mathematics is particularly relevant for task design since it implies that, for the learner, mathematical concepts and the production of signs are mutually constitutive.

Furthermore I use Peirce's notion of mathematical reasoning [2] to deconstruct a CAS–based task into its major components. C. S. Peirce (1839–1914), an American mathematician and founding father of semiotics, proposed that all thinking is performed upon signs of some kind or other, imagined or perceived. Peirce regarded signs not only as a means of signifying or referring to an object, but also a "means of thought, of understanding, of reasoning and of learning" [2, p. 45] Peirce argued that all deduction and mathematical reasoning involves the **construction** of an appropriate sign or diagram, **experimentation** on this set of signs

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* *MathEduc SC*: C30, N80, U70; *MSC 2000*: 00A35, 97C30, 97C80, 97U70.
*Keywords and phrases***:** computer algebra system, reasoning, semiotic approach, task design.

through manipulations and transformations of signs (written, spoken or imagined), and observation (which of necessity includes **interpretation**) of the transformed set of signs.

COMPONENTS OF TASKS

Peirce's categorization of mathematical reasoning is particularly useful for isolating key components of mathematical tasks and examining their implications for teaching and learning. In the context of a CAS–based task, these activities assume a very particular form and function.

I illustrate the illuminating power of Peirce's categories by using these categories to critique a CAS–based task, which I adapted from an undergraduate mathematics textbook [3] in 2007. This task was part of an assignment given to a class of 202 first-year university mathematics students of varying abilities and computer experience. The assignment was intended to introduce students to the concept of the Maclaurin polynomial before the students had been introduced to the concept in regular mathematics lectures. The purpose of the particular task (see Figure 1) was to help students understand the notion of an interval on which the Maclaurin polynomial approximated the original function by a certain amount (e.g., 0.1 units) and how to find this interval.

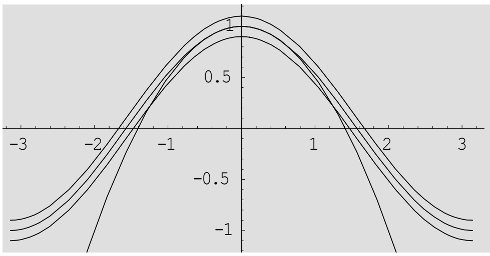

Figure 1. Maclaurin polynomial task

In a previous task, students were required to generate the second order Maclaurin polynomial $p(x) = 1 - ½\,x^2$ and to plot a graph of $p(x)$ and $f(x) = \cos x$ on the same set of axes (see [4] for further details).

Determine the values of x for which the quadratic approximation $p(x)$ found previously is accurate to $f(x)$ within 0.1. [Hint: Graph the functions, $f(x) = \cos x$, $p(x)$, $y1 = \cos x + 0.1$ and $y2 = \cos x - 0.1$ on a common screen.]

Construction of sign

At some point, most CAS–based tasks require students to construct a suitable mathematical signifier (a set of inscriptions, symbols). This may be a representation of a mathematical object, such as a graph or the definition of a function, or it may be a statement concerning a procedure, such solving an equation.

With regard to the exemplar task, my expectation was that students would represent the situation graphically as per the hint and use this result to gain epistemological access to the problem. However only 58% of the class (117 of 202 students) successfully constructed an 'illuminating' plot of the four graphs (often after much trial and error) by using an adequate domain for the graphs; 26% of the class generated the graphs using a less than optimum domain (although 9% of the class were still able to solve the problem despite domain difficulty); 17% were unable to produce a reasonable plot. Thus approximately 34% of this class were unable to construct appropriate signifiers for this task.

Although we may think that first–year mathematics students should be able to easily find an optimum domain (or window), this did not happen for approximately one-third of my class. Possibly this was due to a lack of familiarity with *Mathematica* (these students had a one–hour *Mathematica* tutorial every two weeks and 13% of the class were not computer literate on entry to university). In this pedagogical context, I suggest that the task would have better achieved its intended aim (regarding the notion of an approximating polynomial) had I explicitly suggested a reasonable window. Certainly in pedagogical contexts where students are better rehearsed in using CAS to sketch graphs, such an explicit instruction may not be necessary. This example illustrates see how a small change in the task design may have powerful implications for the quality of the learning that might result.

A brief mention of syntactical intricacies which students encounter when using CAS is necessary. For example, different usages of the equal sign imply different types of equivalence in *Mathematica.* To define a function, $f(x)$, as say, $\cos x$, one enters $f[x\_]:=$ Cos[x]; to solve an equation, one enters, say: [Solve[$f[x] == g[x], x$]; to define a constant, one enters, say, Area = $\pi * r^2$ where r has been defined as radius. Being able to distinguish between these different uses of the equal sign requires a type of mathematical awareness different from the mathematical awareness required in paper-and-pencil work. Inter alia, it requires an understanding of different types of equivalence. Such rarefied knowledge is a crucial part of transforming the CAS into a tool for learning. (Transforming CAS into a tool for learning is one aspect of the process of instrumental genesis, a theoretical framework used by many educators in the CAS world. See [5] for elaboration of this framework.)

Clearly adequate design of CAS tasks requires awareness of the type of hybrid knowledge (mathematical or syntactical) required of a CAS user to construct appropriate mathematical signs. Where necessary the task design should include guidance on such construction. Or the complexity of the different uses of signs can be exploited in the task design. For example, a task may require a student to explicitly distinguish between different uses of the equal sign and to articulate the mathematical significance of that distinction.

Experimentation

Most non-trivial mathematical tasks involve some form of experimentation. I mention two forms of experimentation relevant to CAS here.

CAS tasks resulting in conjectures: One of the most important uses of CAS as a tool of learning is students' use of this resource to generate examples of a particular mathematical construct. From these many examples a student may generate hypotheses of generalised mathematical statements that then need to be proved, probably using paper and pencil. This sort of task is important in several respects. It allows students to move from the particular (various examples) to the general. Furthermore students need to be aware that finding patterns, a common feature of reform type mathematics, is not sufficient to establish mathematical truth. For mathematical certainty, deductive proof (not an inductive conjecture) is required.

Pragmatic versus epistemic values of experimentation: One of the much espoused virtues of CAS is that it can be used as a tool for outsourcing processing power; that is, one can use CAS to perform time consuming and tedious tasks. However this use of CAS as a tool for computation is not without consequences. The French CAS school (see [5]) differentiates between activities that have a pragmatic value and activities that have an epistemic value. Pragmatic values concern the efficiency or productive potential of certain mathematical activities; epistemic values concern the extent to which mathematical activities contribute to an understanding of mathematical objects.

For example, one can use *Mathematica* to find the zeros of polynomial equation $x^5 - 10x^3 + 9x = -4x^4 + 40x^2 - 36$. One enters [Solve $x^5 - 10x^3 + 9x = -4x^4 + 40x^2 - 36, x$]. The computer outputs the zeros in question in the form: $\{\{x\to-4\},\{x\to-3\},\{x\to-1\},\{x\to 1\},\{x\to 3\}\}$. Clearly this is a very efficient way of solving the equation–it has high pragmatic value, and in, say, a mathematical modelling task it may well be a productive use of CAS (enabling the student to devote her intellectual energies to the modelling task). However, the epistemic value of hand solving an equation will largely be lost in the CAS environment. To solve $x^5 - 10x^3 + 9x = -4x^4 + 40x^2 - 36$ by hand, one might write: $f(x) = x^5 - 10x^3 + 9x + 4x^4 - 40x^2 + 36$. One can then use the Factor Theorem[1] together with polynomial division (and a lot of time) to find that $f(1) = 0, f(-1) = 0, f(3) = 0, f(-3) = 0$ and $f(-4) = 0$. Hence $x = -4$ or $x = -3$ or $x = -1$ or $x = 1$ or $x = 3$. The epistemic value of such hand work is high; through this experimentation the student may gain insight into the nature of roots of a polynomial, the Factor Theorem, and so on.

Similar to the French school argument about the technical–conceptual cut [5] and as illustrated by the above example, procedural knowledge is intricately linked to conceptual knowledge [6]. As much as one may regard the execution of certain procedures (such as solving a fifth-order polynomial) tedious and mechanical, one may gain conceptual knowledge about the properties and features of the mathematical construct. Thus, although the use of CAS may free the student from cumbersome activities, its thoughtless use may contribute to the underdevelopment of certain concepts, which are often learnt in tandem with procedural activities.

In my exemplar task, the experimentation with CAS was intended to have both a pragmatic aspect and an epistemic aspect. I expected students to use CAS to gen-

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[1] Factor theorem: The number c is a root of the polynomial function $y = p(x)$ if and only if $x - c$ is a factor of $p(x)$.

erate a visual picture of the mathematical situation more easily and accurately than with paper and pencil (the pragmatic aspect). One reason why this was not always realised might be partly traced back to many students' problems with constructing an appropriate window of graphs, as discussed above. The epistemic aspect involved the students seeing a visual representation of the second-degree Maclaurin polynomial together with $\cos x$. This was expected to give students insight into a Maclaurin polynomial—an insight not easily achieved through by-hand work.

With respect to task design, I could have attempted to increase the epistemic value of the task by withholding the hint (thereby requiring students to actively reflect on the mathematical situation). But I suspect that, without the hint, many students would have been unable to move forward and much epistemic value would have been lost for these students. This illustrates how the provision or withholding of certain information in a task statement may enable or inhibit experimentation and profoundly affect the epistemic value of the task.

Interpretation

In doing any mathematical task, a learner needs to interpret various signs. The interpretation of CAS–based signs is often different from the interpretation of traditional mathematics signs. This interpretation may involve a CAS user simply being aware of certain conventions in the output format of the CAS. In the example mentioned above, the user needs to know that output '$x \to 4$' is equivalent to the paper-and-pencil line '$x = 4$'. Or it may involve a more subtle mathematical awareness. For example, CAS may produce the same simplified expression for two non-equivalent algebraic expressions. Asking, say, *Mathematica* to simplify $(x^2 - 1)/(x - 1)$ yields the output $x + 1$ with no mention of the restriction, $x \neq 1$, and so the user needs to interpret the output with a mathematically critical eye.

Interpretation of graphical representations in a computer environment also entails its own challenges and opportunities. Students' work with my Maclaurin polynomial illustrates this dichotomy. In their work with the Maclaurin polynomial task, two students use an (illuminating) plot of the four graphs that they had generated to help them interpret the task. Using visual inspection they make the crucial observation that the polynomial, $p(x)$, intersects only $\cos x - 0.1$. In this example, the CAS–generated graphs provide the students with epistemological access to the mathematics of the task. Indeed, one student exclaims: "**I think I can see. I can see what is going on**. 'Cause this, no. It's the graph of $\cos x - 0.1$ (*pointing to screen*). It goes here (*pointing to graphs of* $\cos x - 0.1$.) It intersects with the graph of $p(x)$ and it goes here". The pair of students then proceeds to correctly solve the task. In solving tasks, some CAS features generate challenges for learning, whereas some CAS features provide opportunities for learning.

But 'seeing' does not necessarily lead to an appropriate interpretation of the mathematical content. Specifically, although many students were able to identify the points of intersection in the Maclaurin polynomial example, only approximately one-third of them interpreted the points of intersection of the two graphs as the endpoints of an interval (which is what they were required to do). For these students it was as if use of CAS becomes the aim of the task rather than a means to solving the problem given in the task. This is our challenge.

Thus we see, as with traditional mathematics, the interpretation of CAS–based tasks may be unexpected. The extent to which a teacher should guide students in their interpretations depends very much on the particular task and the teacher's intentions. This is an area worthy of further research.

CONCLUSION

I conclude this talk by returning to its beginning: CAS–based tasks require construction and interpretation of signs with different rules for production and interpretation as compared to paper-and-pencil math, although the underlying meaning structure of mathematics is usually the same in both cases. This meaning structure is both reflected by and constituted through the activities with signs, be they paper-and-pencil or CAS–based. In particular, the construction of appropriate CAS–based signs is often problematic for students, and the level of guidance in this regard is a challenge for the teacher. At the same time, complexities around the construction of signs may open up new possibilities for task design. Furthermore the teacher needs to be aware how the provision or withholding of certain information in a task statement may enable or inhibit experimentation or interpretation. This is surely an ongoing challenge and requires further research.

Hopefully this talk has suggested new ways of looking at the structure of tasks and demonstrated how a small change in the task design can effect a large change in the sort of learning that takes place.

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IMPROVING CAS: CRITICAL AREAS AND ISSUES *

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Abstract. Although Gjone [1] and Kadijevich [2] consider several areas and issues that are critical to CAS improvement, they do not examine substititions with CAS or CAS compatibility. Building upon CAS improvement and a long Austrian experience in CAS-supported mathematics, this paper looks back at relating different representation and problems with autosimplification discussed in [1, 2], and examines substitutions with CAS and CAS compatibility. CAS manufacturers should provide a full linkage of different representations, control over autosimplification, a flexible approach to substitutions, and some CAS compatibility.

CRITICAL AREAS

Relating different representations

Despite its importance [1], a full working linkage between different representations has not been realized to our satisfaction until now. The TI-Nspire concept shows some good aspects, but as Kadijevich illustrates [2], it does not work in all directions.

Many teachers complain that organizing a linkage between applications is sometimes too difficult for average pupils (and for not highly motivated teachers, too). GeoGebra (www.geogebra.org) offers a good linkage between representations as shown in Figure 1 (following the example given by Kadijevich [2, Screenshot 5]), but, at present, only numerical solutions are returned (a future version is expected to return results in the algebraic mode).

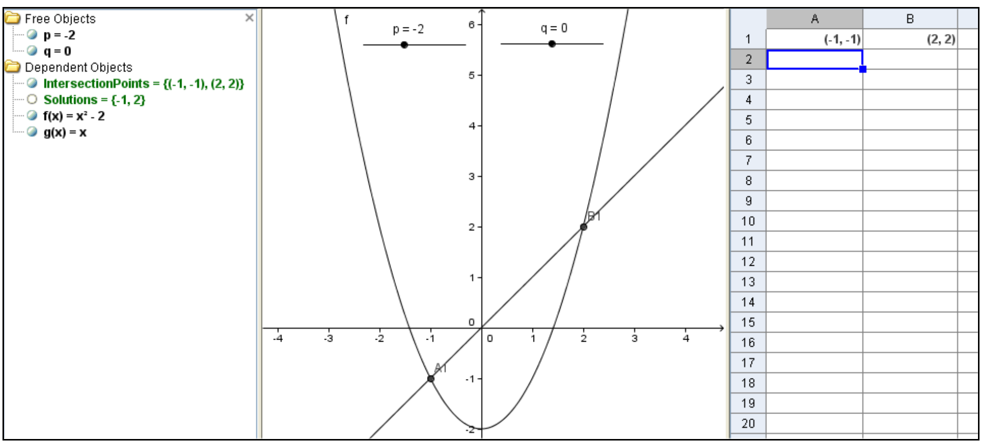

Figure 1. Linking representations with GeoGebra

* *MathEduc SC*: N84, P54, U74; *MSC 2000*: 00A35, 97-04, 97C80, 97U70. *Keywords and phrases*: computer algebra system, software design, software evaluation, upper secondary.

Equivalence and Substitutions

Gjone [1] emphasizes the importance of determining equivalence of algebraic expressions. This is doubtlessly a basic activity in mathematics and can be supported to a high degree by CAS—only supported but not completely done in many cases. Such a limitation applies to trigonometric identities, such as $\sin\frac{\pi}{18}\cdot\cos\frac{\pi}{9}\cdot\cos\frac{2\pi}{9}$, and expressions containing roots, such as $\sqrt{\sqrt[3]{9}+6\cdot\sqrt[3]{3}+9}$ [3, 4]. See Figure 2.

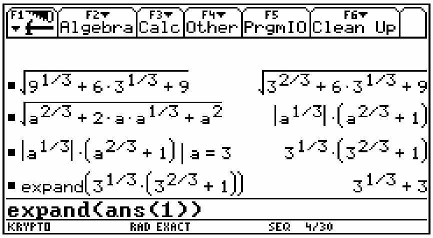

Figure 2. Nested roots on the TI

By applying trigonometric identities or substitutions performed by the user manually (or by using special commands as tExpand), DERIVE and a TI-handheld can eventually find the exact value of the first trigonometric expression (1/8 = 0.125), whereas the TIs cannot find that value for the second expression with radicals. Also, to solve differential equation $y'+y\cdot\cos x=\sin 2x$, $\sin 2x = 2\sin x\cos x$ is needed before the TI´s command *desolve* is able to return the solution.

Another basic activity is substitution, especially at a lower level of mathematics education, and to apply it, the TI tools have the so called with operator. Figure 3 shows that in Derive one can choose between substituting only one occurrence of an expression (from #1 to #2) or all of them (from #1 to #3). This can be done by selecting the sub-expression with the mouse or with arrow keys (e.g., ↓, →, ↓ and → (together with Shift) lead to $a+b$). [5] This has a nice equivalent in WxMaxima with command *substpart* (e.g., *substpart*(v, expr, 3, 2, 2), which substitutes sub-expression of *expr* located at 3, 2, 2 with *v* [6]).

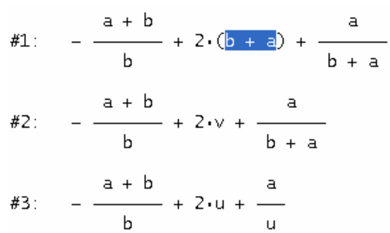

Figure 3. Two substitutions in Derive

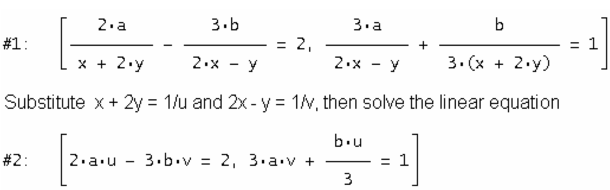

Figure 4. Substitution of subexpressions in Derive

Figure 4 shows application of sub-expression substitution for applying a technique for solving the system of equations. Having determined *u* and *v*, back-substitution is to be used.

CRITICAL ISSUES

Autosimplification

As Kadijevich underscores [2], the user cannot turn auto(matic)simplification on and off, which, by using skilfully created user-defined functions, may help him/her to obtain right answers in subtle cases. There are many occasions when the teacher is not so happy with the fact that CAS performs some work that should be done by students. Figure 5 (left) displays how the Voyage 200 reacts after the user enters an expression containing fractions. Assume that the teacher wants the students to reproduce the paper-and-pen calculation using their CAS (to check their results and simultaneously improve their CAS competence). He/she then prefers a DERIVE way displayed at Figure 5 (right). A group of teachers including the author of this paper asked the TI-developers to add a “NON-Autosimplification” mode that leaves the input as it is. But, unfortunately, until now we could not agree on a common way to achieve this.

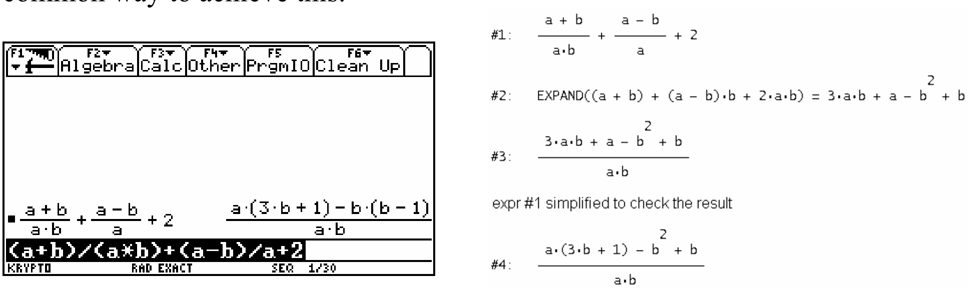

Figure 5. To autosimplify or not to autosimplify?

CAS Compatibility

It is clear that CAS documentation has to be improved [2], but moreover the most used commands and functions should look similar. The various systems may differ in extra features and commands, which might be attractive for some teachers. Figure 6 displays how creating a simple sequence is done in three CA systems used in the Austrian schools. Although the reader may recognize the three systems, a request for a “Dictionary for Computer Algebra Systems” seems quite appropriate. Note that the same problems occur by comparing the results of many commands (including, for example, the domain of underlying functions), which was emphasized in [1].

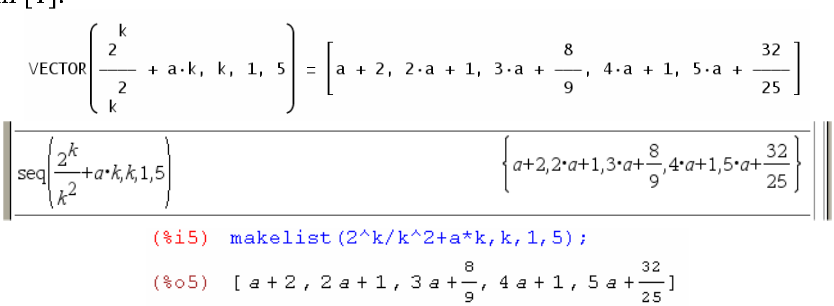

Figure 6. Several ways to define a sequence

Because modern day students are accustomed to using several software at the same time, they should be encouraged to use different CAS [7]. This approach reinforces the issue of CAS compatibility, which is not addressed at present For example, some basic CAS command may be understood and executed correctly in different CAS (see the first two lines in Figure 7), but is is more likely that the effect of the copy-and-paste action from one CAS to the other, even if displayed as intended, will not be recognized by the other CAS (see the line in Figure 7 indicated by the arrow).

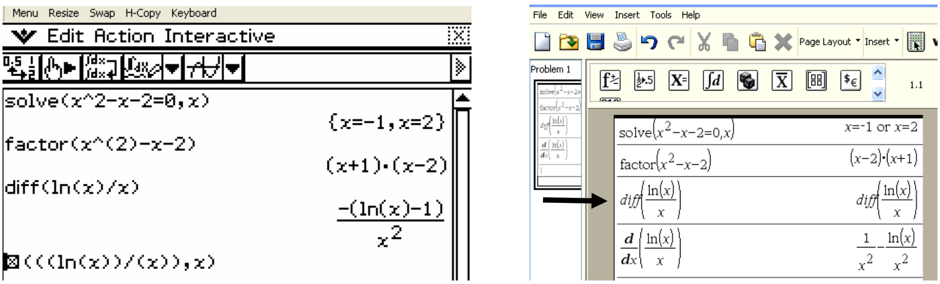

Figure 7. Multitasking with Casio ClassPad and TI-Nspire CAS (provided by Dj. Kadijevich)

CLOSING REMARKS

CAS manufacturers should provide a full linkage of different representation forms and a control over autosimplification. Substitutions should be made as flexible as possible by using pointing devices or appropriate commands that recognize the expression structure. Finally, because of its learning benefits and practical constraints (e.g. a need to use other CAS), some CAS compatibility should be attained.

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CAS-BASED TASK FRAMEWORKS AND LINKING MULTIPLE REPRESENTATIONS *

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Abstract. The establishment of theoretical frameworks to inform the design of CAS-based tasks provides a recommended structure for the intended role of such tasks in classroom practice. The implementation and use of CAS-based tasks by teachers and students introduces complexities related to the coordination of CAS and paper-and-pencil mathematical work and the critical role of classroom discourse that is focused on bolstering students' reflection and reconciliation of the combined work. Additional consideration is given to the salient feature of a CAS machine—symbolic manipulation—and the expanded possibilities afforded when CAS-based tasks foster flexible use of symbolic and graphical representations.

INTRODUCTION

The issues presented in Margot Berger's paper [1] are centered on a proposed framework for CAS-based task design, and the respective roles of the teacher and student in the process of using such tasks. This paper elaborates on the issues raised in regard to the proposed CAS-based task design principles and the importance of making connections between graphical and symbolic representations, one of the pedagogical affordances of a CAS machine.

TASK DESIGN PRINCIPLES

In other frameworks that articulate the role of CAS in task situations, the connections between CAS and by-hand work are given a more prominent focus. For example, the work of Kieran and Saldanha [2] corroborates the French theory (see [3]) by underlining the co-development of conceptual and technical knowledge through mathematical activity that involves reconciling CAS and paper-and-pencil work, reflecting on these reconciliations, and proving the generalized results. This theory extends the construction and interpretation components proposed in [1] in two ways.

First, in the construction arena, the complexity of blending paper-and-pencil and CAS-based work should involve more explicit attention to the role of reflection and discussion in classroom practice. In a task where CAS and paper-and-pencil are used in tandem, Kieran and Saldanha [2] suggest that purposeful reflection and discussion will bolster the power of CAS as an instructional tool, and also push students' conceptions of algebra beyond what is possible without such a blended use of tools and guided classroom discourse. Hence the construction component of CAS-based task design should not ignore strategic pedagogical moves that focus students' mathematical thinking on relationships between the semiotic forms generated by-hand and those generated by machine.

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* *MathEduc SC*: C30, N80, U70; *MSC 2000*: 00A35, 97C30, 97C80, 97U70.

Keywords and phrases: computer algebra system, representation, task design.

In the context of a CAS-based task on factoring with high school algebra students, Kieran and Saldanha's work [2] extends the interpretation component in the realm of reconciling differences in by-hand and CAS output. These researchers argue that, "the coupling of students' attention to the forms produced by the CAS in comparison to their own productions … helped constitute these forms and their structure as objects of explicit reflection and classroom discussion" (p. 408). The claim that students' interpretations of CAS forms become explicit in this view offers a variation to the suggestion by Berger [1] that "the extent to which a teacher should guide students in their interpretations depends very much on the particular task and the teacher's intensions" (p. 6). In particular, the reconciliation of by-hand work with the sometimes unexpected CAS output should be an explicit feature in the design of each CAS-based task, and included in the structure of classroom discourse, highlighting an extension of the interpretation component of the framework.

LINKING MULTIPLE REPRESENTATIONS VIA CAS

Berger [1] exemplified her espoused framework with a task that utilized the graphing capabilities of a CAS with no attention to the symbolic capabilities that are typically characteristic affordances of a CAS machine. The de-emphasis on the more salient affordances of a CAS begs the question of when does a task actually require the use of CAS? Related to this issue is the important pedagogical responsibility of developing in students the habit of mind of using technology judiciously, which might involve the coordination of multiple tools and multiple representations of a single mathematical object. The black box/white box approach [4] of the Maclaurin polynomial task [1] could lead to greater experimentation with the mathematical concepts involved, especially if a symbolic approach is more closely integrated to the graphical representation.

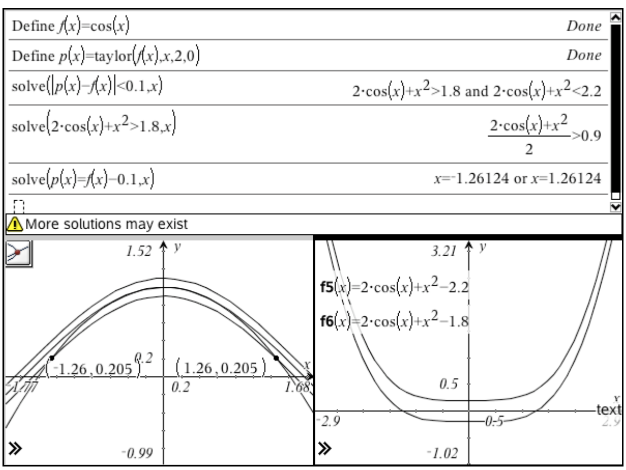

Figure 1: Symbolic and graphical solution approaches

As Lokar posits in [5], research supports that making connections among multiple representations has been positively linked to students' understanding of mathematical concepts. For an example, refer to the CAS screen (Figure 1) that was created on TI-Nspire™ CAS Teacher Edition computer software [6]. Students could attempt to solve for *x* when the difference in *p(x)* and *f(x)* is less than *0.1*, as in line 3. A non-informative attempt in solving such a solution symbolically (line 4) prompts one to consider a graphical solution (lower-right graph). While the zeros of *f5(x)* yield a correct solution to the problem as a range of *x* values, the connection to the original problem may not be apparent. This is where the graphical approach suggested by Berger [1] comes into play. To facilitate the issue of finding an appropriate graphing window, the TI-Nspire allows the capability for the user to drag the axes and the screen to stretch and reposition the graph until the desired viewing screen is achieved. The graphical intersection points correspond to the symbolic solution in line 5. The warning that more solutions may exist is a perfect opportunity for students to reflect on why such a prompt would be given. Asking students to interpret why the given solution set is not comprehensive brings the graphical representation (lower left) back to the forefront of discussion. I have found that in working with pre-service secondary mathematics teachers this process of moving between representations promotes various solution approaches but requires guidance in both task design and classroom discourse that focuses students' thinking on reconciling the differences in each representation.

CLOSING REMARKS

The integration of multiple theoretical frameworks for CAS-based task design (e.g., [1, 2]) may lead to a synergistic outcome of a more versatile and powerful structure for such tasks. In particular, the complexities involved in the coordination of paper-and-pencil and CAS tools should be paid explicit attention to in the task design, and in the design of pedagogical strategies that the teacher employs to facilitate reflective discussion with students. A productive area of future research would be to examine students' ability to flexibly use graphical and symbolic representations in a CAS-based task designed to exemplify the aforementioned theoretical framework.

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THE USE OF CAS IN SCHOOL MATHEMATICS: POSSIBILITIES AND LIMITATIONS *

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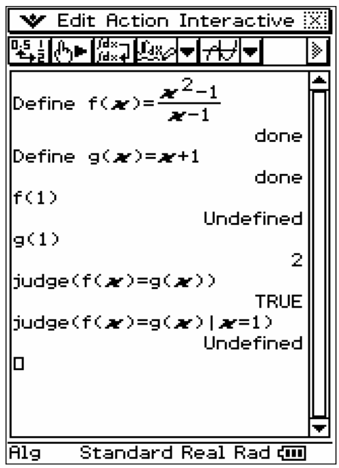
Abstract. By using examples from upper secondary mathematics given for the Casio ClassPad tool, this presentation deals with determining equivalence of algebraic expressions, relating different representations, and making Computer Algebra Systems (CAS) techniques transparent and congruent with their paper-and-pencil versions. For each of these three topics, the presentation underlines its importance for CAS-based research, shows what best can be achieved with this tool at present, and summarizes limitations that should be addressed in a future version of the tool. In order to do mathematics with CAS in a better way, both CAS features and their use need to be improved.

INTRODUCTION

The use of CAS has been an important issue in secondary school mathematics for more than ten years (see www.lkl.ac.uk/research/came/). From the beginning of using such tools in the 1980s, there has been a development in improving their possibilities. By applying the perspective of working mathematically, this paper presents examples of possibilities and limitations of calculator/software *Casio ClassPad* (www.classpad.org). These examples are given for three topics that are particularly relevant to CAS-based school mathematics. These topics are determining equivalence of algebraic expressions, relating different representations, and making CAS techniques transparent and congruent with their paper-and-pencil versions.



Screenshot 1

DETERMINING EQUIVALENCE OF ALGEBRAIC EXPRESSIONS

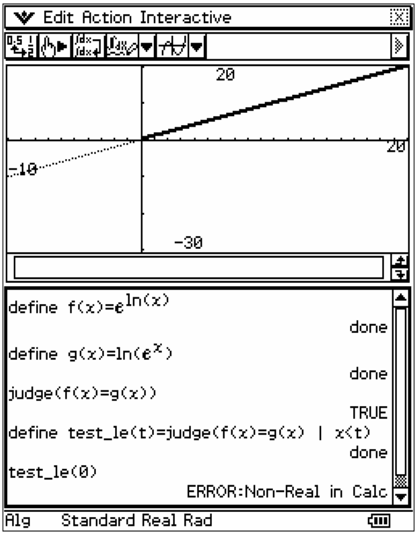
Topic importance

A large part of doing mathematics deals with transforming expressions to fit imposed requirements. Considering equivalent expressions is hence an important topic in mathematics teaching. Reasoning about equivalent algebraic expressions is an important research area not only in traditional mathematics education, but also in CAS-based mathematics education [1], where due to CAS limitations, students should skillfully use various CAS commands (e.g. the equality test and solve) to find out whether two expressions are equivalent.

* *MathEduc SC*: N84, P54, U74; *MSC 2000*: 00A35, 97-04, 97C80, 97U70. *Keywords and phrases***:** computer algebra system, software design, software evaluation, upper secondary.

Tool affordances

There is a ClassPad command named **judge** that is able to provide the answer to the question of equivalence in most cases. But, as shown on Screenshot 1, despite a restriction of domain (with "|"command), the use of judge may be of little value with special subtle cases. The strength of command judge may be improved as presented on Screenshot 2. We find here that the user, perhaps believing **f(x)** and **g(x)** are not equivalent, applies function **test_le** (defined by him/her or other user) and gets message "ERROR: Non-Real in Calc".



Screenshot 2

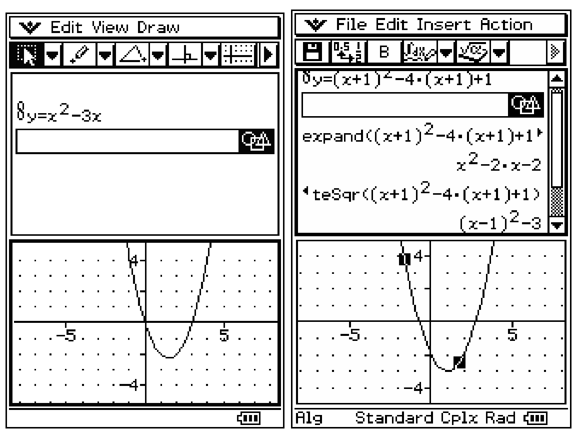
Issues to improve

According to [2], equivalence of expressions is not a decidable problem. In other words, due to theoretical reasons, equivalence of expressions cannot be verified in all cases even by ideal CAS. Because of that, subtle user-defined functions (created by teacher or able students) need to be used. This approach is not well supported by ClassPad at present as, for example, functions cannot be defined in several lines, or by using several commands in a line separated by ":". Also, there may be problems when using *if-then-else* statement implemented as **piecewise** function (see the appendix).

RELATING DIFFERENT REPRESENTATIONS

Topic importance

Using different representations of the same mathematical entity is one of the most important aspects of doing mathematics. A flexible dealing with different representations is usually seen as a step towards understanding of mathematical entities. As CAS are essentially representation tools, the question of creating, using and relating different representations is highly relevant to CAS-based mathematics education [3].

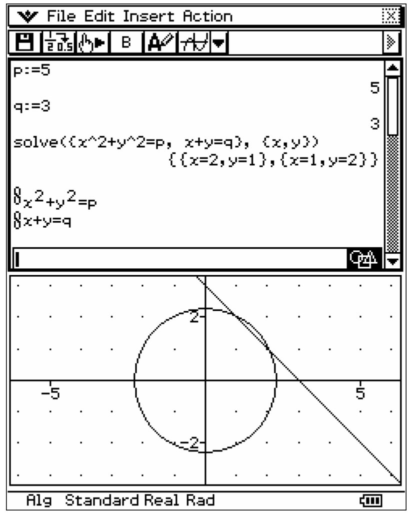


Screenshot 3

Tool affordances

Screenshot 3 present a ClassPad feature, *Geometry Link,* that links an algebraic representation of a function with its graphical representation in such a way that any change in one representation is reflected in the other. This feature is particularly useful in examining graphs of elementary functions under certain transformations (e.g. $y=x^2$ vs. $y=(x-1)^2-5$).

Screenshot 4 presents how two geometry links can be used in solving systems of equations, where solutions (2, 1) and (1, 2) can be recognized as the points of intersection of the two graphs.



Screenshot 4

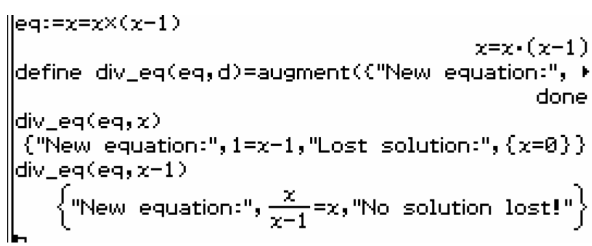
Issues to improve

The Geometry Link feature is limited, however. Indeed, if we change the values of parameters p and q (Screenshot 4), the solutions of a new system will be found, but the two graphs will not change. Also, a geometry link with a function on a restricted domain (e.g. $y=x^2 \mid x > 0$) will wrongly result in the graph of that function on its full domain. These two examples evidence that ClassPad applications (e.g. e-Activity and Geometry) need to be better linked in a future version of the tool.

MAKING CAS TECHNIQUES TRANSPARENT AND CONGRUENT WITH THEIR PAPER-AND-PENCIL VERSIONS

Topic importance

Paper-and-pencil techniques may not be reflected in their CAS versions (e.g. factorization of $x^9 - 1$). Also, CAS technique (i.e. the way in which CAS solves a class of problems) is not open to its user (hence the requirement: turn a back box of CAS technique into a white box or a grey one. (Term technique is used in the sense of the French school [4].). The success in doing mathematics with CAS is thus considerably constrained by the transparency of CAS technique as well as the congruence of that technique with its paper-and-pencil version [5]. Without reducing these constraints, CAS cannot be used in a functional, strategic and pedagogical way as properly required by [6].



Screenshot 5 (provided by Dj. Kadijevich)

Tool affordances

In order to have equation and inequality solving with CAS that is transparent and mirrors the usual paper-and-pencil work, several functions can be defined (by teacher or able students), explained to students, and used by students to help them be more aware of the solving processes. An example is given in Screenshot 5 (we divided both sides of the initial equation by x and then by $x-1$ and obtained proper answers).

Like the Casio FX2.0 CAS calculator, ClassPad also sometimes accepts syntax where parentheses are left un-closed (e.g. **solve**($x^3-1=0$ returns $\{x=1\}$). Some students like to use this time-saving, but counter-mathematical, feature.

Issues to improve

In order to improve shortcomings concerning issues of transparency and congruence, better CAS commands and appropriate user-defined functions are needed. These functions can be defined with ClassPad in a limited way as described in the end of the section on expressions equivalence. Also, some useful functions known to ClassPad (e.g. **element**({1, 2, 3}, 2) and **completeSqr**(x^2-4x)) cannot be found in the ClassPad manuals. Finally, some functions may still work in a strange way (e.g. **mode**({a, b, a}) is a, **mode**({true}) is TRUE, whereas **mode**({true, false, true}) is {TRUE, FALSE, TRUE}). All these make the work with user-defined functions a hard and somewhat frustrating job.

CLOSING REMARKS

Good user-defined functions are crucial to improving CAS. This requires CAS manufacturers to provide better conditions for the development of user-defined functions, taking into account critical issues given in this paper.

Contrary to paper-and-pencil mathematics, "defining a function is required before an expression such as $f(x)$ or $f(2)$ can be used." [4, p. 68]. Further CAS-based research may focus on the work with user-defined functions.

ACKNOWLEDGEMENT

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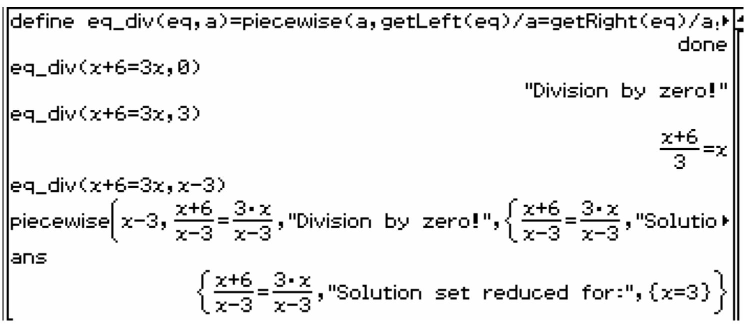
APPENDIX – USING PIECEWISE FUNCTION

Piecewise(0, 1, 2, 3) returns "2", **piecewise**(2, 1, 2, 3) returns "1", whereas **piecewise**(*x*, 1, 2, 3) return "3" although "3" should be returned in each of these three cases as the first argument of this if-then-else-error function is not a relation. By exploiting this behavior in a constructive way, define the following function:

```
define eq_div(eq,a)=piecewise(a, getLeft(eq)/a=getRight(eq)/a, "Division
by 0!", piecewise(judge(getLeft(eq)=getRight(eq) | solve(a)),
{getLeft(eq)/a=getRight(eq)/a, "Solution set reduced for:", solve(a)},
getLeft(eq)/a=getRight(eq)/a))
```

Screenshot 6 evidences that this function cannot be executed directly because Casio ClassPad returns **piecewise**(*x*–3, …), which is executed by using system variable **ans** containing this intermediate answer.



Screenshot 6

CRITICAL ISSUES OF IMPROVING COMPUTER ALGEBRA SYSTEMS *

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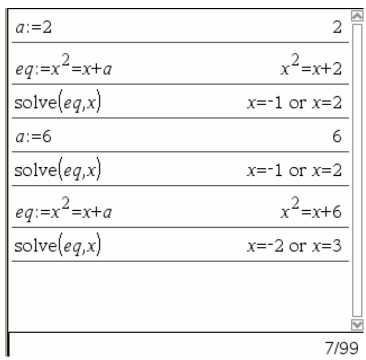
Abstract. In order to improve Computer Algebra System (CAS), there is a need not only for better built-in features to be implemented, but also for better conditions for creating and using user-defined functions to be provided. By using the TI-Nspire CAS tool and examples from high school mathematics, this paper examines the following critical issues of improving CAS: updatable lines with commands and functions, controllable execution of commands and functions, richer tool documentation, stronger links between different representations, richer opportunities for creating and using user-defined functions, and customizable user interface. These critical issues should be adequately addressed in a future version of the tool.

INTRODUCTION

Gjone [1] examines the possibilities and limitations of the Casio ClassPad tool for critical topics of determining equivalence of algebraic expressions, relating different representations, and making CAS techniques transparent and congruent with their paper-and-pencil versions. He finds that in order to improve CAS, not only do better built-in features need to be implemented, but also better conditions for creating and using user-defined functions need to be provided. Acknowledging the relevance of these requirements, this contribution examines critical issues related to these requirements. The issues deal with updatable lines with commands and functions, controllable execution of commands and functions, richer tool documentation, stronger links between different representations, richer opportunities for creating and using user-defined functions, and customizable user interface. These six issues are exemplified in the next section by using examples from high school mathematics that are mostly worked out on the TI-Nspire CAS tool (www.ti-nspire.com).



Screenshot 1

SIX CRITICAL ISSUES

Updatable lines with commands and functions

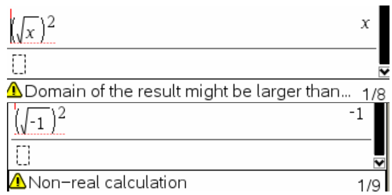
Contrary to potential user expectations, TI-Nspire CAS does not support editing expressions in the calculator history. It is only possible to copy all or part of an expression from that history, paste the material to the entry line and then edit the material, if needed. As the result of such a limited input, the outcome of the new calculation in the entry line does not affect the outcomes of calculations in lines above that line. For example, when parameter a is made equal to 6 (Screenshot 1,

* *MathEduc SC*: N84, P54, U74; *MSC 2000*: 00A35, 97-04, 97C80, 97U70. *Keywords and phrases*: computer algebra system, software design, software evaluation, upper secondary.

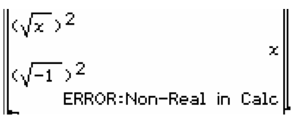
line 4 as counted from the bottom of the screen), the value of equation *eq*, defined as $x^2 = x + a$ and then displayed in line 6 using $a = 2$ from line 7, is not updated to $x^2 = x + 6$, and therefore the zeros in question are not updated in line 3 to -2 and 3. In order to obtain the desired results, two lines ($eq := x^2 = x + a$, solve(*eq*, *x*)) are copied and executed again. [Note: Although Casio ClassPad behaves in the expected way (with respect to on-screen editing, and subsequent updating of existing results in lines below the updated line), the content of its geometry links are not updated, however (see [1], Screenshot 4)].

Controllable execution of commands and functions

Knowing that TI-Nspire CAS automatically simplifies expressions when working with complex numbers [2], it is easy to understand why $\left(\sqrt{x}\right)^2$ simplifies to x and why the produced value of $\left(\sqrt{-1}\right)^2$ is -1. A good thing is that each of these results—obviously a wrong one in the domain of real numbers—is coupled with an appropriate warning in real mode. These warnings are "Domain of the result might be larger than domain of the input" and "Non-real calculation", respectively (see Screenshot 2). However, TI-Nspire CAS does not save the warnings when it saves a document containing the results.



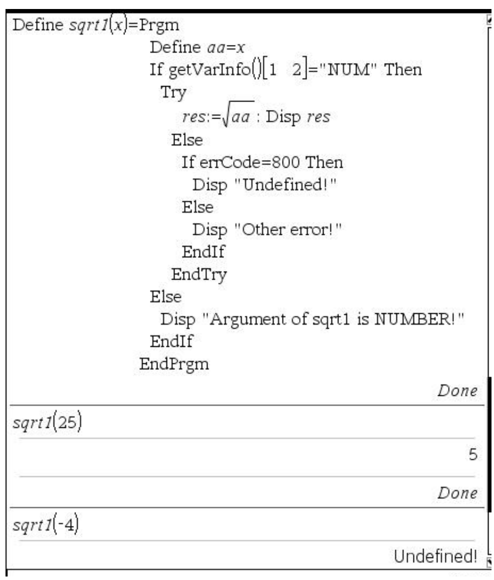
Screenshot 2



Screenshot 3

The user can choose to work with only real numbers (by using option File/Settings/ Document Settings…), but the results are the two reported previously not the following two: "x for $x \geq 0$" and "Undefined". (In real mode, TI-Nspire CAS returns "ERROR: Non-real result" for $\sqrt{-1}$, whereas it displays "undef" for log(0).) As regards the automatic simplification, the user cannot turn it on and off to obtain right answers in subtle cases by using skilfully created user-defined functions. Screenshot 3 presents answers to the two questions by Casio ClassPad.

Controllable execution of commands includes managing errors. Although TI-Nspire CAS enables managing errors (see Screenshot 4), the



Screenshot 4

TI-Nspire CAS Reference Guide does not provide codes of the warning messages. Note that ClassPad 330 User's Guide contains tables with error messages and warning messages, but codes for these messages and system variables handling them are not given, and managing errors is thus not possible.

Richer tool documentation

Extending the conversation about managing errors, this third critical issue deals with improperly explained CAS features and unexplained CAS features.

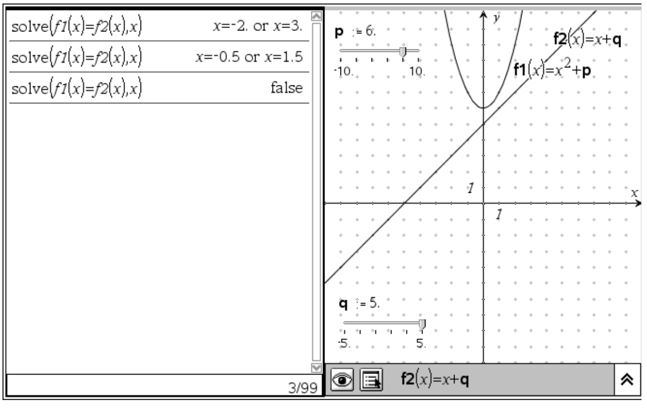
Some commands or functions may give answers that differ from those described in the official documentation. Consider, for example, an implementation of if-then-else-error function: the Casio ClassPad function piecewise wrongly returns 1 for **piecewise**(13, 1, 2, 3), whereas TI-Nspire CAS correctly returns 3 for **when**(13, 1, 2, 3). Furthermore, Casio ClassPad correctly returns "No solution" for both **solve**($x^2 = 4$, a) and **solve**($x^2 = 4x$, a), whereas TI-Nspire CAS wrongly returns $x = 2$ or $x = -2$ for **solve**($x^2 = 4$, a) and even $x = 2\sqrt{x}$ and $x \geq 0$ or $x = -2\sqrt{x}$ and $x \geq 0$ for **solve**($x^2 = 4x$, a).

Official manuals may not describe several useful commands and functions that can be used. Try, for example, **completeSqr**($x^2 - 2x - 8$) on Casio ClassPad. (Such a command or function on TI-Nspire CAS has not been discovered.) Although hiding some features may be a defendable business policy, disclosing some of these features would help us improve our use of the tool.

The previous section examines the question of managing errors from the programmers' point of view. An ability to manage errors depends on detailed information regarding errors, which should appear in CAS official manuals.

Stronger links between different representations

Despite a powerful ClassPad feature called Geometry Link, links between algebraic and geometry representations in two windows (i.e. applications) are not strong because the content of geometry links cannot be updated automatically (see [1]). Links between two applications are also not strong for TI-Nspire CAS because, as Screenshot 5 displays, new values for two parameters p and q (assigned with sliders) require another execution of the same solve command. Better links may simply require an update command that would update the content

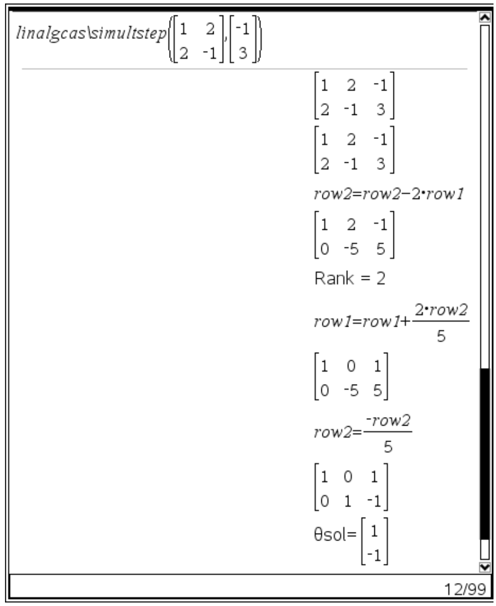


Screenshot 5

of other applications (here the Calculator application) to reflect the changes in the active application (here the Graphs & Geometry application).

Richer opportunities for creating and using user-defined functions

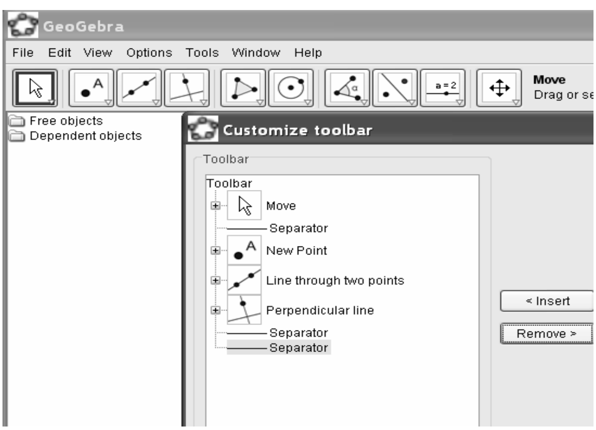
Opportunities for creating and using user-defined functions should include creating functions in a program-like fashion (with managing errors) and using these functions from libraries of functions. Screenshot 4 presents an example of such a created function. A step-by-step solution of system of two linear equations, $x + 2y = -1$ and $2x - y = 3$, is given in Screenshot 6. This solution is found by the application of function *simultstep* from library *linalgcas* (i.e. document *linalgcas* is stored in a designated library folder such as My Documents\TI-Nspire\MyLib). Both Screenshots 4 and 6 present the opportunities in question offered by TI-Nspire CAS, which should nevertheless include managing warning messages, better control of command and function executions, and stronger links between different representations in different applications. As noted in [1], opportunities for creating user-defined functions are very limited in Casio ClassPad.



Screenshot 6

Customizable user interface

An educationally oriented CAS such as Casio ClassPad and TI-Nspire CAS may be used for several years during secondary and tertiary education. In early years of this use, just small subsets of CAS commands and functions would be utilized. As CAS affordances would grow as the mathematics known by its user grows, CAS should have a customizable user interface. Screenshot 7 presents such an interface available in GeoGebra, which is dynamic geome-



Screenshot 7

try software including, among others things, some CAS features (see www.geogebra.org) This kind of interface, not available in Casio ClassPad and TI-Nspire at present, should deal both with built-in and user-made affordances.

CLOSING REMARKS

In order to have CAS that is a pedagogical tool, CAS should make its techniques transparent and congruent with their paper-and-pencil versions [3,4], distribute computational tasks both to CAS and its user [3], provide step-by-step solutions [5], take care about subtleties such surplus or missing solutions in solving equations [6], update the outcomes reflecting changes in linked entities (cf. [3]), and have a customizable user interface [3]. By taking an integrated perspective of these requirements, this paper examined six critical issues of improving CAS that deal with both built-in and user-made features. Further research may examine these issues in more detail to help CAS manufacturers make appropriate decisions concerning future CAS development. By using these and other relevant critical issues, further research may also study the process of instrumentalization [7] regarding the development of user-made features.

ACKNOWLEDGEMENT

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CAS-BASED TASK REQUIREMENTS AND CRITICAL ACTIVITIES IN COMPLETING THEM *

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Abstract. Having in mind essential features of mathematics, appropriate CAS-based tasks should require representing, operating and interpreting with CAS. Critical activities in completing these tasks may be found in transitions from representing to operating, and from operating to interpreting. These critical activities—related to mathematics, technology or both—are, for example, using CAS to apply transformations of represented objects, and relating CAS results with mathematical questions aimed to be answered.

INTRODUCTION

According to Berger [1], critical design features of CAS-based tasks basically deal with constructing signs, using these signs, and interpreting the outcome of their use. She discusses these three features through consideration of an approximation of a function by a polynomial with a certain precision, suggesting that help in solving CAS-based tasks may be needed with respect to each of the features. In this reaction to [1] we wish to underline two important yet neglected issues. First, the three design features respectively deal with three kinds of mathematization: representing, operating and interpreting [2], each influencing the others. Second, by extrapolating from [3], critical activities in completing these kinds of mathematization with CAS should be found in transitions from representing to operating, and from operating to interpreting (intepreting with CAS without representing, and some operating with it, seems quite rare). In this account "CAS" does not denote just a symbolic algebra functionality but an integrated environment involving several tools (applications) supporting the work with several representations of mathematical entities.

REQUIREMENTS

Three basic activities in mathematics are representing, operating and interpreting [2]. As a result, mathematics can be viewed as the science of doing and relating these three kinds of mathematization, where using technology allows more time for representing and interpreting as well as for reflecting on the three mathematizations and their relations.

In addition to operating, technology can be used for representing (e.g., defining a function or generating a graph) and interpreting (e.g., visualizing an unfamiliar solution or searching a library of solutions of related problems). Because of that,

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Keywords and phrases: computer algebra system, reasoning, learning activities.

CAS-based tasks should require the use of CAS for each of the three mathematizations. Easier tasks should focus on only one of them with CAS gradually adding the other(s) with focus both on them and their relations.[1] For example, we may start with representing the given problem situation with a piece-wise linear function, or with interpreting the solution of a corresponding equation (where the outcomes of representing and operating are provided). Then we may relate representing and operating to illustrate how a change in representing (e.g., from algebraic representation to graphical representation of piece-wise linear functions) may add complexity to or reduce it from operating. In general, using different representations may improve problem solving as well as understanding of the underlying mathematics as demonstrated in [4]. Although Berger [1] presents a useful framework to guide the design of CAS-based tasks, she does not describe what CAS affords with respect of each of the three mathematizations and relations among them.

CRITICAL ACTIVITIES

Critical activities in a transition from one modelling stage to the next are examined by Galbraith and Stillman [3]. As regards the transition from real world problem statement to mathematical model supported by technology, these researchers recognize the following nine critical activities:

1. Identifying dependent and independent variables for inclusion in algebraic model,
2. Realizing that independent variable must be uniquely defined,
3. Representing elements mathematically so formulae can be applied,
4. Making relevant assumptions,
5. Choosing technology/mathematical tables to enable calculation,
6. Choosing technology to automate application of formulae to multiple cases,
7. Choosing technology to produce graphical representation of model,
8. Choosing to use technology to verify algebraic equation,
9. Perceiving a graph can be used on function graphers but not data plotters to verify an algebraic equation. (See [3, p. 147].)

These activities describe not only what should happen when a particular transition is achieved with a success, but also what blockages are likely to cause a failure in that transition.

Modelling is one approach to doing mathematics. Having in mind the generality of the representing-operating-interpreting framework, this framework can be found in the modelling stages. Also, as underlined in the previous part, it is important to relate different kinds of mathematizations with CAS. Because of that, the presented approach of Galbraith and Stillman [3] may be extrapolated to transitions within the representing-operating-interpreting framework. In doing so, we tried to recog-

[1] Even tasks focusing on one critical aspect may be hard for students. Examples include tasks that ask one to “Sketch the graph of the given function on its whole domain”, and more demanding tasks that request one to “Find the function whose graph is given."

nize critical activities in transitions from representing to operating, and from operating to interpreting. Mostly relying on our experience in using CAS as a learning tool, we recognized that main critical activities in question may be (list is neither exaustive nor final):[2]

- **From representing to operating**: Using CAS to apply transformations of represented objects; Using CAS to verify a representation that will be used for calculation or interpretation; Using CAS to calculate.
- **From operating to interpreting**: Relating CAS results with mathematical questions to be answered; Preparing for interpreting through obtaining additional results with CAS; Preparing for interpreting through integrating all CAS results obtained.

Consider, for example, a task that requires solving the system of equations, $x + y = a$ and $x^2 + y^2 = 25$, in terms of parameter *a.* Assume that the user decided to use a parameter and two equations, chose to use an algebraic tool and a geometry tool, represented the parameter and the two equations with the algebraic tool, and represented the two equations with the geometry tool. The six critical activities listed above may be found in the following activities (the order of three within each transition is not fixed):

1. Compare the outcomes in the geometry window for different values of the parameter;
2. Produce a table of values for a relation in question and match it to the correponding graph produced by the geometry tool;
3. For concrete values of the parameter, find the solution of the system with the algebraic tool;
4. Recognize that the distance of the origin from the line (found first for concrete values of a) is related to the solvability of the system;
5. In terms of parameter *a,* find the solution of the system in question and the distance of the origin from line $x + y = a$ (a user-defined function may be used for the latter);
6. Integrate the results obtain under 5 having in mind related results obtained under 1 and 3.

In her account on designing CAS tasks, Berger [1] suggests that help in solving CAS-based tasks may be needed with respect to each of the three kinds of mathematization. Guidance for help in solving such tasks (i.e., scaffolding) – missing in her account – may profit from the presented critical activities, enabling teacher and researcher to manage better the design and use of CAS-based tasks.

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[2] Main crticial activities in transition from task statement to representing may be: Identifying objects to be used (functions, equations, inequalities, or others); Choosing CAS tool(s) to represent these objects; Representing identified objects with CAS.

CLOSING REMARKS

Appropriate CAS-based tasks should require representing, operating and interpreting with CAS, which would dynamically relate relevant conceptual and procedural knowledge (cf. [4]). Critical activities in completing these tasks, found in the two transitions mentioned above, are related to mathematics, technology or both. As students tend not to coordinate use of mathematics and use of relevant e-tools [5], particular attention should be paid to critical activities related to both mathematics and CAS combining exact mathematical language and math-jargon of technology [6].

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THE INTEGRATION OF INNOVATIVE CAS SOFTWARE: THEORETICAL FRAMEWORKS AND ISSUES RELATED TO THE TEACHER *

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Abstract. This paper introduces some results of ReMath related to Casyopée (a CAS learning environment incorporating a Dynamic Geometry window) regarding the theoretical frameworks involved and the teacher, and presents on-going research work about the dissemination of Casyopée beyond the ring of expert teachers. Preliminary findings indicate mid-adopters take approximately six months to appropriate the software, mid-adopters struggle as they attempt to integrate the tool's constraints, and expert teachers serve as mediators between mid-adopters and researchers.

INTRODUCTION

The rationales and history of the Casyopée project at an earlier stage have been explicated by Lagrange [1]. The Casyopée team brings together teachers and researchers to take up the challenge of using Computer Algebra Systems (CAS) to teach about functions at upper secondary level, consistent with recent curricula. The question addressed by the Casyopée team deals with the possibility of developing and disseminating a CAS environment that could help students to freely experiment, choosing their own way of solving and proving. Thanks to the ReMath European project, the Casyopée team was able to progress toward this goal. This paper introduces the Casyopée software, reports on the results related to Casyopée in the ReMath project, especially regarding the teacher, and presents ongoing research work about the dissemination of Casyopée. [For further information, please visit http://jb.lagrange.free.fr/site]

CASYOPÉE IN THE REMATH PROJECT

The ReMath project addresses the task of integrating theoretical frames on mathematical learning with digital technologies at the European level. A specific set of six dynamic digital artefacts (DDA) has been developed, reflecting the diversity of representations provided by ICT tools. Seven teams from four countries are involved in this project. Casyopée is one of the six DDA, and its use has been investigated in Italy and France. In order to explain the software's functionalities, we now elaborate on the type of problem whose resolution can take advantage of Casyopée, and how. Figure 1 contains an example.

Consider a triangle ABC. Find a rectangle MNPQ with M and Q on [AB], N on [AC], P on [BC] and with the maximum area

Figure 1. A 'generic' optimization problem.

Constructing a generic triangle ABC in the geometrical window can be done after creating parameters in the symbolic (CAS) window. For instance, the points can be A(-a;0), B(0;b) and C(c;0), with a, b and c being three positive parameters. Then

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one can create a free point M on the segment [oA] (o being the origin) and the rectangle can be constructed using dynamic geometry capabilities.

In the Geometric Calculation tab (Figure 2), one can calculate the area of rectangle MNPQ and then define an independent variable. Numerical values of calculations and of the variable are displayed dynamically when the user moves free points. The user can then explore the co-dependency between these values. If this calculation depends properly on the variable it can be exported into the symbolic window; that is to say that, thanks to its CAS capabilities, Casyopée computes the domain and algebraic expression of the resulting function.

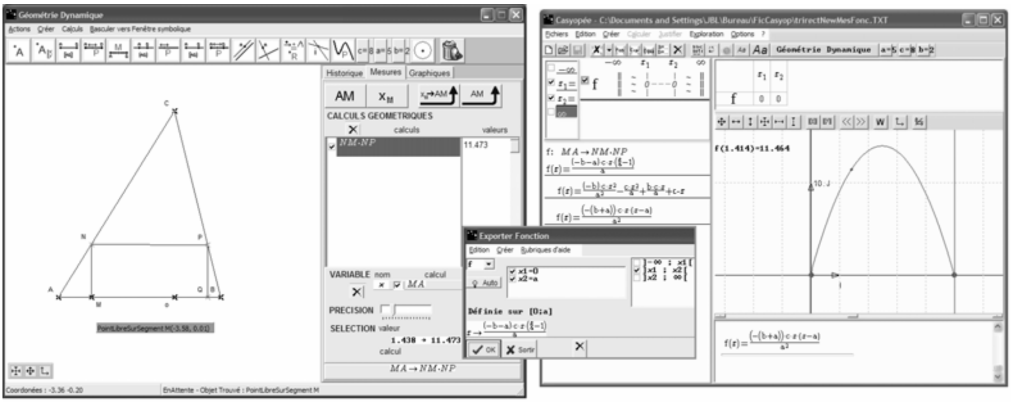

Figure 2. Casyopée's symbolic and IG windows and the exportation form.

After exporting into the symbolic window, one can work on various algebraic expressions of the function and on graphs. For instance, one can use properties of parabolas or algebraic transformations or Casyopée's CAS functionality of symbolic derivation to find the answer to the question. One can also use the graph of the function to conjecture about the maximum area.

TAKING INTO ACCOUNT THE TEACHER IN CLASSROOM SCENARIOS

We report here on the Italian experiment [2], carried out at grade 12 over 11 school hours. This experiment highlights the role of the teacher for a successful implementation of Casyopée's potentialities under the inspiration of the Theory of Semiotic Mediation [3]. The designed educational goals were (a) to foster the evolution of students' personal meanings towards the mathematical meanings of function as co-variation, the notions of variable, and domain of a variable, and (b) to foster the evolution of students' personal meanings towards mathematical meanings related to the algebraic modelling of geometrical situations.

Students had previously received some formal instruction on the notions of variable, function and graph of a function, and on its graphical representation. The aim was to mediate and weave those meanings in the more global frame of modelling. Hence, the pedagogical plan was not constructed with the purpose of helping students become able to use Casyopée for accomplishing given tasks, but instead to foster the students' consciousness-raising of the mathematical meanings at stake. It was structured in cycles entailing: students' pair or small group activity with Casyopée for accomplishing a task, students' personal rethinking of the class activity (through the request to students of producing individual reports on that activity), and class discussion orchestrated by the teacher.

The familiarization session was designed as a set of tasks providing students with an overview of Casyopée features and guiding them to observe and reflect upon the "effects" of their interaction with the tool itself. Besides familiarization, the designed activities consisted of coping with "complex" optimization problems formulated in a geometrical setting and posed in generic terms, as in Figure 1. The aim was to elaborate on those problems so as to reveal and unravel the complexity and put into evidence step by step the specific mathematical meanings at stake.

The teacher's delicate role is to guide students to unravel such complexity and to make the targeted mathematical meanings emerge. The teacher has to achieve this objective by way of class discussions. The development of a class discussion cannot be completely foreseen a priori, it should be designed starting from the analysis of students' actual activity with Casyopée and of the reports they produce, and it would still depend on extemporary stimuli. Nevertheless in the design the Italian team tried to anticipate possible development of the pedagogical plan and to plan some kind of possible canvas for the teachers for managing class discussions. In contrast, the French team, influenced by other theoretical frameworks, like the theory of didactical situations [4] and the instrumental approach, designed its experiment through a careful choice of mathematical tasks, overlooking the definition of the teacher's actions. For the French team, it was very useful to learn from the Italian team how to pay more attention to design of the teacher's action.

DISSEMINATING CASYOPÉE TO 'SECOND ADOPTERS'

We[1] are now studying the dissemination of Casyopée. A first assumption is that a CAS tool designed by researchers to have a high potential, like Casyopée, will not necessarily be easy and welcome for teachers not involved in the project, even if it has been developed in close connection with expert teachers. Research about dynamic geometry environments [5] shows that there is a wide gap between researchers' visions about the contribution of digital tools and ordinary teachers' expectations with regard to technology. Our aim is then to investigate ways of bridging this gap. Another central assumption is that researchers have to consider the dissemination of their production as a research work in itself, creating communities involving teachers-users and researchers to produce varied resources associated with the use of the software.

[1] Nguyen Chi Thanh, from the Hanoi Pedagogical University, is working with the author in a postdoctoral contract supported by the AUF (Agence Universitaire pour la Francophonie).

In these communities all teachers are not to be considered at the same level. We propose considering “first-adaptors,” or “experts,” to be teachers that have been constantly associated with the project development and think of the integration of Casyopée as a natural process. We propose to consider also “mid-adopters,” teachers who are interested by using technology in the classroom, but were not associated with Casyopée’s development. We expect that these teachers will be primarily interested in easy-to-achieve and close-to-curriculum applications of Casyopée, and sensitive to conditions of its successful integration. We also expect that these teachers, more than the experts, will be concerned by problems and constraints such as those related to the time required to implement technologies in their classes, to curriculum requirements, to overall school organisation issues, and to training needs. All other teachers at upper secondary level, as potential users of Casyopée, form a third layer.

It seems that transition from experts to mid-adopters is crucial. If we can “break the barrier” around experts, we will have made a very significant step towards dissemination, especially by learning about “second adopters’” needs and their paths toward integration. Our idea is to use “scenarios” (classroom activity plans) as a means for communicating between layers: the elaboration and trials of scenarios by second adopters will be first a way of communicating between experts and mid-adopters: mid-adopters will propose uses corresponding to their needs and ask experts for advice and support. The scenarios will be designed to be accessible to all teachers and to be a way to communicate between the second and third layers.

At a theoretical level we first consider these “scenarios” as “boundary objects” between researchers and teachers and between teachers in different layers. Their importance has been noted by researchers concerned with interaction between communities [6]. These objects do not carry meaning with them, instead meaning is recreated, in action. We cannot assume that the meanings we build into any artefact are transparent to teachers. Teachers construct their own meanings, influenced by their past experiences and beliefs as well as their interactions with the objects. Mutual negotiation and meaning-construction thus should be established as the norm for both sides of the boundary, rather than the preserve of one protagonist.

The second theoretical notion is instrumental genesis [7]. The potential role of digital tools cannot be expected to be transparent and if they are to be integrated in a significant form into mathematics classrooms, an understanding of how to engender the process of instrumental genesis is crucial. In working with teachers, the instrumental genesis process is particularly complex since artefacts become instruments in their didactical practices as well as in their mathematical practices.

METHODOLOGY

Expert teachers

Two teachers (named here Bernard and Xavier) were especially involved in the cyclical process of specifying functionalities for a CAS software, contributing to the software development, and experimenting with their classes. Crucial steps in the project, especially the decision to develop a software environment around a symbolic kernel and to append a Dynamic Geometry window, were undertaken as a consequence of dissatisfactions they expressed after classroom experimentations. The design was driven by their careful attention to the consistency of the computer representations of the mathematical objects and to the design of Casyopée as a tool helping students rather than constraining them.

Bernard and Xavier and their students feel comfortable with Casyopée in spite of its constant evolution. Casyopée's complexity is generally not resented by their students because of the careful introduction these teachers provide, making the connections between Casyopée's objects and the mathematical objects.

Mid-adopters

In the same region of Rennes where Bernard and Xavier teach, a group of six teachers had been constituted in the IREM (Institute for Research in Mathematics Teaching) to experiment with the Interactive White Board (IWB). During three years they had experimented with the use of software packages (IG and CAS) on the IWB, and they were keen to enter a new project. For us they were good candidates to be "mid-adopters": they were convinced that technology can support mathematical teaching and learning, they were relatively experienced in the classroom use of technology but were not, like Bernard and Xavier, involved in Casyopée's history. In the present project their role is to prepare and experiment under the supervision of Bernard and Xavier scenarios that will be disseminated to mathematics teachers together with Casyopée on the professional digital workspace for teachers in Brittany.

PRELIMINARY FINDINGS

At the present stage of this research study, we can only outline some features that we found relevant or surprising. As we are analysing the data (records of meeting sessions and scenarios proposed and experimented by second adopters), the author will be able to present a more complete set of findings at the CAME meeting.

Six months were necessary for mid-adopters to appropriate the software in order to be comfortable in the classroom. In contrast to the experts, accustomed to explaining to students that some difficulties or malfunctioning may occur, the mid-adopters were very worried about presenting to their classes software that is "not totally finished." It took also time for these teachers to understand how the characteristics of the symbolic kernel influences Casyopée's operation.

Because of their experience with other software, the mid-adopters found often difficult to integrate Casyopée's constraints. For instance, seemingly drawing on their work with numerical dynamic geometry systems, they created indistinctly as free points the "generic points" defining the figure (the triangle's vertices in figure 1) and the "moving point" defining the variable elements of the figure (a vertex of the rectangle in figure 1). The consequence is that the "exportation process" did not work, because too many free points were involved in the variable elements.

It is a surprise that the mid-adopters never considered scenarios about modelling a geometrical dependency like in the ReMath project. Our hypothesis is that they already have strategies to handle these problems in the classroom, based on numerical dynamic geometry systems. In contrast with the experts, they do not see the limitations of these strategies and Casyopée's advantages. The scenarios they created show how they were attracted by other functionalities of Casyopée that we did not expect to be first considered.

Another feature is the respective roles of the researchers and of the expert teachers. Researchers very often need to answer second adopters' questions regarding the software and the aims of the development and thus need to be very active in the meetings. The expert teachers generally act as mediators, explaining specific positions. Their contribution is essential for researchers to understand mid-adopters' reactions.

CONCLUSION

The acceptability of CAS by teachers is central in Casyopée's design. The ReMath project experiment shows that, beyond software design, a crucial issue is to take into account the teacher's activity in classroom scenarios. Preliminary findings of the dissemination ongoing project indicate several ways in which mid adopters' view of Casyopée contrasts with experts'. Further work will help to understand the differences between layers of teachers and how this understanding can be a basis for building communities of teachers and researchers to support dissemination.

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SOME ISSUES ON DESIGNING TASKS FOR CAS CLASSROOMS *

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Abstract. When designing CAS tasks we should not look on a CAS as an isolated tool but in connection with other tools. This paper examines how with an appropriate selection of tools can circumvent several shortcomings of CAS tasks. It requires CAS tasks to be adaptable to the needs of their users with respect to procedural and conceptual issues. Finally, it underscores the idea that designing and evaluating CAS tasks should address the whole process of their design, usage and modification.

INTRODUCTION

In her paper Berger [1] gave us valuable insight into some of the core problems arising when using CAS tasks in mathematical education. In my reaction I try to address the issues Berger raised, but from a slightly different perspective. I use the obstacles Berger observed and the shortcomings mentioned to illustrate some currently available technical possibilities that might enable CAS task designers to circumvent the aforementioned problems.

APPROPRIATE TOOLS

As mentioned in [2], we are in the process of changing the view of mathematical knowledge from a hierarchical structure to a flexible network structure. The construction of tasks should reflect that change. When a CAS-based task is evaluated, it is too often observed within a closed environment, from the perspective of a specific CAS used with the task. Several problems that have been observed in these evaluations could be seen in a completely different light if another tool were used. Berger mentioned in her paper "the task would have better achieved its intended aim … had I explicitly suggested a reasonable window" [1, p. 3]. But is a hint really necessary? Would it perhaps be better to use another tool, and allow students to find the appropriate window on their own?

Task designers all too often exhibit the one-size-fits-all syndrome. Namely they wish to stick with the same tool at any cost. Some amazing tricks can be seen; features are exploited in unusual ways (in, for example, Derive, GeoGebra, or Mathematica) or tricky instructions are provided, everything so as to stay within the same environment, the same program.

It is often argued that students feel more comfortable if they work with the same program all the time. However, when the behavior of modern day students (the so called NetGeneration students) is observed, it is obvious that multitasking is not problematic to them. Different programs are used at the same time; students are switching between windows, gadgets, and tool. Information is collected from mul-

* *MathEduc SC*: N80, U30, U70; *MSC 2000*: 00A35, 97C80, 97U30, 97U70.
Keywords and phrases: computer algebra system, learning resources, task design, teachers.

tiple sources. Students, while communicating visually, are quick to respond and expect the same from others. They are able to "shift their attention rapidly from one task to another" [3, p. 2.5]. When designing CAS-tasks we should exhibit the appropriate use of tools. CAS (or a particular CAS) is not always the best choice. Students should be taught how to choose the most suitable tool.

Instead of arguing about the best way to cope with problems that a specific tool introduces into a certain task, we should design the task itself in such a way that students are not merely allowed to but are actively encouraged to use different tools and to mix tools. Research has clearly established that ease of using different representations of various mathematical objects is perhaps one of the most beneficial influences in developing conceptual knowledge in mathematics. There is now a need for research that shows how appropriate use of different tools either promotes or hinders the students' progress.

NOTATION

A lot of technological tools disregard certain notation issues such as: *n* denoting a whole number, *log* being the decimal logarithm. Some programs assume that *x* is always used as the independent variable in functions, and * as the multiplication sign. Mathematics has a complex set of notations that are in common use. The notations used at each stage of schooling are carefully chosen. CAS often introduce new symbols or expose students to symbols they would not normally be exposed to at a certain stage. On the other hand, students often adapt more quickly to different notation than we think they can [4]. Consistent with the above mentioned observation states that students need to be prepared for situations as yet unknown, exposing students to different notation and a different set of symbols is almost essential.

Berger argues, "Being able to distinguish between these different usages of the equal sign requires a type of mathematical awareness different from the mathematical awareness required in paper-and-pencil work" [1, p. 3]. Is this true? Are we not merely trusting that, in a given paper-and-pencil situation, students do know that "=" as used in $f(x) = x + 2$ is a different "=" than that used in $x + 2 = 7 - x$? Is it perhaps not true that the rigidity of CAS tools might prove to be beneficial for the students and help them become more aware of the type of the mathematical object they are dealing with?

Calculate the exact value of the expression $\left(-\frac{2}{3}\right)^{-2} + 0.25^{-\frac{1}{2}}\left(2^{-3} - 1\right)$.

Figure 1. Calculation task from [5]

Mathematical notation is mostly two dimensional. Entering such notation with a keyboard is often a demanding task, even for an experienced mathematician. What about students? For example, the task in Figure 1, when solved with Derive, is almost meaningless. The student does not even need to know the right structure of the expression. The right result is attained even if the student neglects the parentheses around (-1/2) in the exponent.

Perhaps the main problem as regards mathematical notation in CAS setting is a less than adequate user interface most CAS (as well as other tools) possesses. CAS manufacturers have tried different approaches. But these various approaches raise even more questions about the task design. Whether the interface used influences the students' perception of the task is just one of them. A more practical drawback of using a particular approach to solving problems with mathematical notation issues in this rapidly changing technological world is the fact that all these approaches are usually close form solutions. So they cannot be easily transferred to different CAS.

Hopefully, new breakthroughs in interface technology (e.g., surface computing, handwriting, speech recognition) will provide solutions that will allow us and our students to merge paper and pencil and CAS environments in a simple way.

LEVEL OF GUIDANCE

Berger mentioned, "In particular, the construction of appropriate CAS–based signs is often problematic for students and the level of guidance in this regard is a challenge for the teacher" [1, p. 6]. The importance of tasks being prepared in such a way that they can be adapted according to a particular didactical situation should be mentioned yet again. If a task designer overcomes the desire "to stay within the same environment", it is technically possible to design a task in a much more flexible way. The challenge between the two different approaches Berger mentioned (i.e., increasing epistemic value by withholding a hint, providing the hint so students might move forward) can be overcome if certain (and now technically possible) options are exploited.

CAS tasks are all too frequently designed in a way that closely mimics the learning path used in a paper-and-pencil environment. I strongly believe this is not necessary. For example, tasks could be designed in such a way that a link to different task or aid (perhaps a video clip) would be available to students who are unable to progress without a particular hint or that when they were struggling with inappropriate windows of graphs. Hyperlinks, an appropriate user interface with dynamically open content, are one technically easy means that can be used in task preparation.

When I was thinking about Berger's example, a question that simply cannot be avoided occurred to me: did she perhaps want many of the students to achieve too much at one time by having the task goal be "to help students understand the notion of an interval on which the Maclaurin polynomial approximated the original function by a certain amount (e.g. 0.1 units) and how to find this interval" (p. 2)? Should a task be designed in such a way as to provide appropriate fallback? For example, a first task could only deal with the appropriate plotting. If a student was unable to find appropriate windows, he could use a "suggest window" link, something another student would not need to. In this way the epistemic value of the task would be improved.

Namely, instead of speculating the precise ratio of both the procedural and the conceptual approach that would make the task suitable for all students, the fact that this ratio is different for different students and each particular set of circumstances should be considered. Therefore, tasks should be designed in such a way that this ratio can easily be adapted to the needs of the user, be it a teacher or a student.

THE ROLE OF A TEACHER AND REUSING RESOURCES

When we are working with e-teaching materials, we too often find that the authors of such materials, meant for the use of teachers in the teaching process, do not use the opportunities offered by new technologies. All too often the materials are a monolithic block (or at least their main part is), constructed in the way an ordinary book or workbook would be. This demands that the teacher takes them as a whole, precisely in the order they were written in. Is that really necessary? Do all teachers need the same form of resources, do they want to use them in the same order, and do they want all of their students to see the same examples, do the same exercises? Why not use the possibilities that new technologies offer and at the very least give teachers the chance to adapt the materials to their own and their students' needs.

In the design of tasks the role of the teacher is too frequently neglected. The author of the task usually focuses solely on the students. The process of using the task is AUTHOR → TASK ← → STUDENT. The author develops a task and publishes it. A student accesses the task and tries to solve it. He interacts solely with the task. Thus the author is required to incorporate all of the necessary guidance and feedback into the design of the task itself.

But the majority of tasks are used in a different manner. Students are not usually exposed directly to the task, as there is a teacher present in most cases. The teacher serves as a mediator between the task and the student. He chooses an appropriate task. If necessary, he adapts it or provides additional guidance. So the process is really AUTHOR → TASK ←→ TEACHER ←→ STUDENT. The relation TASK ←→ TEACHER, where the teacher adapts the task, is of extreme importance in the teaching process.

Although CAS can be used to create high quality learning resources, such resources are difficult to find at various portals [6]. Several studies (e.g. [7]) also show that teachers use just few of the resources made available. A somewhat surprising fact in itself is that math teachers were especially slow to adopt such materials. Also, there seems to be a decline in the usage of available materials for several reasons. The possibility of task modification is one of the properties teaching materials most often lack and math teachers demand. If teachers have at least the possibility of modifying the teaching material provided, they have a much more positive attitude towards using the particular material. And the teachers’ attitude towards the task used is perhaps the most important part of the usage of ICT in the teaching process. Math teachers, especially those teaching in upper primary and secondary schools, do not like using close form solutions or solutions where the complete didactical situation, in which the task is being used, relies on a particular aspect of CAS; they want to be in control of the whole process [7].

Also, as is the case in Berger's example, the task designer often pays special attention to a particular CAS or a particular didactical situation. As Berger observed, both factors can be of extreme importance and the nature of a task can be drastically changed if the CAS used is different than the one the task designer presupposed. When designing and evaluating a particular task, we should envision the whole process of its design, usage, and modification (see the model illustrated in Figure 2).

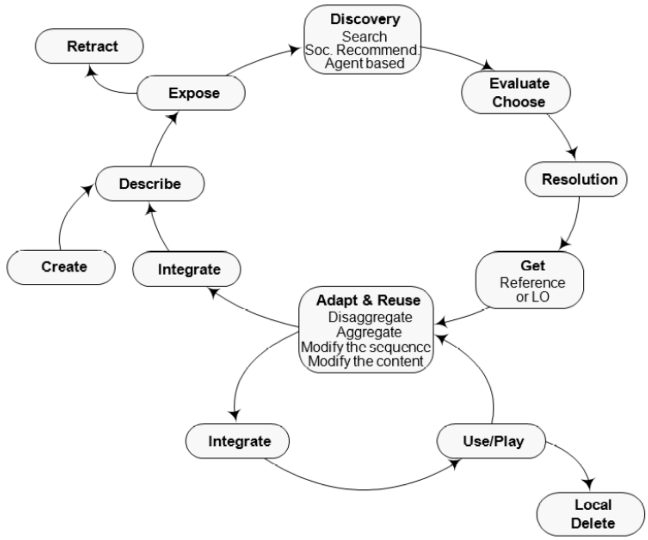

Figure 2. Design/usage/modification process ([8], p. 444)

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TEACHERS EXPERIMENTING WITH TOOLS AND TASKS: FROM FEUERBACH TO APOLLONIUS AND BACK *

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Abstract. Here we discuss issues raised by Lagrange [1] and Mayes [2] regarding the development and implementation of software that links Computer Algebra (CA) and Interactive Geometry (IG) tools. We emphasize that conceptual understanding in using such tools involves (a) flexibilty in transforming representations and (b) integration of experimentation and deduction. We demonstrate our ideas by exploring two famous works on tangent circles utilizing both CA and IG software. One task is based on the classical problem of Apollonius (3rd century BC), namely constructing a circle tangent to three given circles. The other one is Feuerbach's Theorem (discovered in 1822) showing that the Nine-point Circle related to a given triangle is tangent to the inscribed and escribed circles. The problems were presented to teachers in professional development courses on Analytic Geometry.

INTRODUCTION

The main issues raised in Lagrange's paper [1] are: (a) the development of a linked CAS-IG software for learning mathematics [3] and (b) the partnerships of researchers and teachers (especially 'first adopters') in the dissemination of the tool. Mayes [2] addresses the need for flexibility in changing representations while using a CAS-IG tool. In discussing the implementation barriers identified by Lagrange, Mayes emphasizes that teachers need to have experience with how the tool elicits conceptual understanding from students. We believe that conceptual understanding in using tools involves integration of experimentation and deduction. The advent of computer technology opened up opportunities to include experimental mathematics in research and in education [4]. Lagrange mentions only briefly that students conjecture about solutions while exploring algebraic expressions that model geometrical objects. We attempt to facilitate experimental learning and deduction by combining work with two tools, Computer Algebra (CAS) and Interactive Geometry (IG). We found Duval's classification of transformations within (treatment) and between (conversion) representations helpful in the analysis of tasks and tools [5]. The automatic treatments and conversions of representations enabled by the tools imply recognition of the same objects in different representations. In professional development courses we observed that some teachers that are keen on proof tend to switch from experimenting to proving the results with CAS [6]. In a previous study [7] we compared two sub-groups of teachers working on a problem in Analytic Geometry. The sub-group that had only basic experience in CAS-assisted solving outsourced traditional techniques from the tool. The teachers that had extensive experience in using CAS developed new techniques that involved experimentation.

* *MathEduc SC*: B59, C39, U79; *MSC 2000*: 00A35, 97C30, 97C70, 97C80. *Keywords and phrases*: computer algebra system, interactive geometry, representation, deduction, teacher education.

PURPOSE

We wish to learn with the teachers how to develop technology-based curricular materials that enhance both experimentation and proof. At the same time we are interested to identify effective ways for implementing the developed materials in the schools. Naturally teachers' views regarding implementation are affected by their experience in using the tools. Here we present and analyze mathematical–didactical perspectives of utilizing CAS and IG (tools) in exploring two famous works on tangent circles (tasks).

THE MATHEMATICAL PROBLEMS

The Nine-point Circle associated with a given triangle consist of (i) the midpoint of each side of the triangle (D, E, F), (ii) the foot of each altitude (H, I, J), and (iii) the midpoint of the segment of each altitude from its vertex to the orthocenter K (L, M, N) are concyclic. Feuerbach proved that the Nine-point Circle is tangent to the inscribed and escribed circles of the given triangle, known as *Feuerbach's theorem*. Interestingly, the Nine-point Circle is a solution of *Apollonius' tangency problem*. Therefore, we designed a didactic sequence that starts and ends with Feuerbach's work linked by the old problem of Apollonius.

IG is used for introducing the mathematical situation to the teachers as well as for motivating conjectures and attempts to prove them. We begin by constructing the Nine-point Circle, in both environments IG and CAS, for a specific triangle whose vertices are A(-4, 0), B(4, 0), and C(2, 6). After comparing the mathematical steps in the two environments, we ask the teachers to verify Feuerbach's theorem regarding the inscribed and escribed circles. We suggest starting with the IG (see left-hand side of Figure 1). Some teachers are pleased to use the approximately measured length of radii and distance between centres for 'verifying' the tangency of the circles. Other teachers suggest utilizing the algebraic facility of the CAS for reassurance.

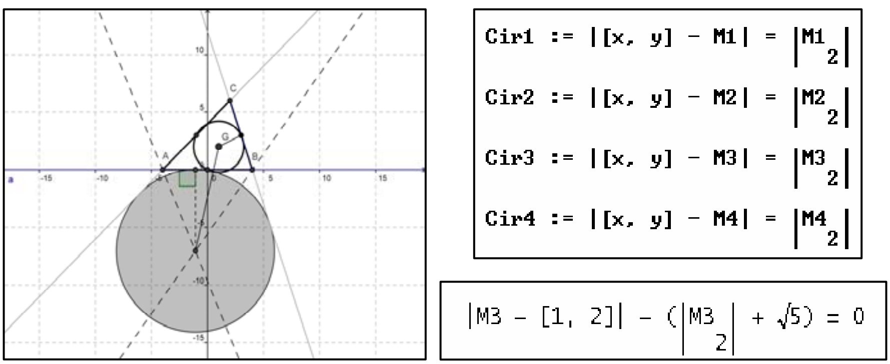
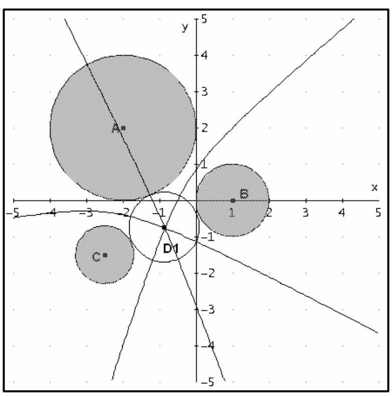


Figure 1. Experimenting with IG and proving with CAS

The construction of the inscribed and escribed circles (by plotting the angle bisector) in CAS is more demanding then in IG. A few teachers realized that the length of the radius of each circle is equal to the y-coordinate of the centre denoted by M (due to the fact the side AB of the triangle lies on the x-axes). In the right-hand side of Figure 1 we see a symbolic representation of the circles and the consequent proof of tangency by comparing the distance between centres and sum/difference of lengths of radii. The fact that the four circles inscribed and escribed in the triangle are tangent to the Nine-point Circle raised our interest in exploring Apollonius' tangency problem with the two tools. The Great Geometer, Apollonius, was interested to construct circles that are tangents to any three objects (points, lines, circles). In the case of three circles up to eight solutions are possible. Searching for approximate solutions by experimenting with IG can be done, but it is cumbersome. In accord with the Greek preference for rigorous Euclidean methods of construction for plane problems, the original problem of Apollonius was to construct a circle tangent to three given circles using only straight-edge and compass. This is equivalent to requiring an algebraic solution involving nothing higher than quadratic equations.

By algebraic methods (using a CAS) we can find a quadratic whose root is the radius r of the unknown circle. We suggested to the teachers an alternative (experimental) approach which takes advantage of the powerful *implicit plotting* facility of the CAS. Three circles are given by their centres A, B, C and radii a, b, c respectively. The discursive representation of a solution circle D with radius d is converted to an algebraic representation $|D - A| = d + a$. By a series of manipulations (treatment) we get three equations of the form $|D - A| - |D - B| = a - b$. These equations are converted by implicit plotting to graphs of three one-branch hyperbolas. The intersection of the three curves (denoted D1) is identified by zooming. D1 is the centre of the desired solution (see left-hand side of Figure 2). By the same procedure we can obtain the other seven solutions (see five solutions in the right-hand side of Figure 2).



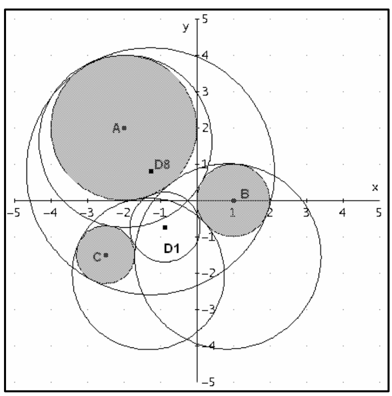


Figure 2. Solution circles to Apollonius' problem

From Apollonius to Feuerbach**:** Feuerbach's theorem provides a proof that a special solution of Apollonius problem for the case of three escribed circles of a triangle is the circle through the midpoints of the sides of the given triangle, that is the Nine-point Circle. This observation links the two famous works.

MAIN ARGUMENTS

Most of the participants in professional development courses were pleased with the experimental methods enabled by IG. The CAS-experienced teachers appreciated the potential of implicit plotting of CAS for experimentation, but also argued for a formal algebraic representation of the results. The cognitive difficulties of conversions of representations as identified by Duval call for didactical consideration. Therefore, it is important to analyze with the teachers examples of transforming discursive, numeric, algebraic, geometrical and graphical registers of representations. Such analysis provides content-based professional development that is, per Mayes, a significant key to dissemination.

CONCLUSIONS

We are aware of the barriers to implementation of the problems and methods presented above. The main reason is that such problems are not part of the current curriculum. A few teachers who participated in the courses chose to present the problems to their students. Some teachers invited team members to present the tasks and the tools in their classes. This gives us opportunities to expose high school students to technology-based problems, as well as to demonstrate to the teachers how the tools elicit conceptual understanding from students. We hope that the students will implement such tools and the tasks as they become teachers.

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TEACHING WITH A CAS-IG TOOL *

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Abstract. By considering a French experiment with Casyopée, this paper examines the theoretical frameworks underpinning its pedagogical plan, the first-adaptor teachers' experience with the tasks and tools, and the challenges of disseminating the tool to mid-adaptor teachers. It emphasizes conceptual development through inquiry into big ideas and professional development driven by sound principles.

INTRODUCTION

The ReMath Project (Lagrange, 2005, in [1]) undertook a significant challenge in developing Casyopée, an integrated CAS-IG tool that allows students to explore algebraic functions as models of geometric dependencies and co-variations. An innovative aspect of Casyopée is the ability to construct geometric figures in the IG component and export geometric dependencies into the CAS component. This feature allows students to view a linked geometric co-variation and algebraic function that models it. Focusing on the interaction of teacher with tool, I address three issues raised in Lagrange's paper: theoretical frameworks underpinning a pedagogical plan, first-adaptor teachers' experiences, and challenges of disseminating the tool to mid-adaptor teachers.

THEORETICAL FRAMEWORKS

The ReMath Project integrates four theoretical frameworks as a foundation for the project, providing a basis for task development and driving the pedagogical plan:

- Setting Notion (Douady, 1986, in [1]): supports the project outcome of switching between geometry and algebra settings and the need for a linked CAS-IG tool;
- Registers of Representations (Duval, 1993, in [1]): supports the goal to develop flexibility in changing registers underlying the concept of function;
- Instrumental Genesis [2]: underlies a pedagogical plan that addresses the teacher's/ student's knowledge of mathematics and the technological tool's capability for multiple representations; and
- Theory of Didactical Situations [3]: casts task design and implementation as a continuous interaction between a subject and milieu in adidactical situations.

These theoretical frameworks provide a strong foundation for ReMath's pedagogical plan. The concept of associated function is introduced as a prerequisite concept, supporting the transition from geometric co-variance to algebraic model. If functions are to be used as models, then the concept of function must be well de-

* *MathEduc SC*: B59, C39, U79; *MSC 2000*: 00A35, 97C30, 97C70, 97C80. *Keywords and phrases***:** computer algebra system, interactive geometry, representation, software dissemination, teacher development.

veloped. In our work with college algebra students [4], we assessed conceptual understanding of function as the ability to shift between process and object interpretations of function and analytic, graphic, and numeric representations of function in a Function Concept Matrix (Figure 1). Translation between some representations is easy for students, such as the numeric to graphic move that requires the lower level skill of plotting points. However, translating from numeric to analytic requires a more sophisticated understanding of modeling data. The need for understanding functions algebraically combined with the need to translate between geometric and algebraic conceptions of co-variation results in a complex task for students. In fact, some teachers might need explicit professional development to develop an adequately deep conceptual understanding to employ Casyopée's IG-CAS link effectively.

Interpretations	**Representations**			
	Verbal/Written	Numeric	Graphic	Analytic
Process				
Object				

Figure 1: Function Concept Matrix

The pedagogical plan addresses understanding of area relationships and prerequisite knowledge for geometric modeling, as well as an opportunity to explore the IG component of Casyopée. As with all powerful mathematical tools for dynamic exploration, there is a steep learning curve. Teachers and students need time to play and discover the capabilities of the tool as instrumental genesis evolves. In using Derive extensively with our college algebra students, we provided detailed directions concerning commands they were to use. Still, students seemed to be overwhelmed at times by the power and flexibility of the tool. I found the CAS and IG components of Casyopée, but not the interface between the two, to be fairly intuitive. Exporting between CAS and IG without significant direct instruction could be a point of confusion for both teachers and students. This raises a broader issue about the complexity and power of CAS and IG packages. While having a package like Derive or the Geometer’s Sketchpad that provides an open workspace and numerous tools allows the teacher to address an almost endless variety of tasks, it often overwhelms both teacher and student. Only first-adaptors, using Lagrange's term, seem to make the effort to teach with such a tool without extensive professional development. How do we address the issue of powerful tools that are either used minimally (in terms of time or level of sophistication) by teachers? One approach is to develop micro-tools such as Java based applets. Micro-tools focus on one conceptual idea and provide only the features needed to address that idea. This reduces the confusion and time to impact in the classroom for the teacher and student. Casyopée is a powerful tool that may benefit from reducing some of its features to focus on the concept of geometric co-variations modeled algebraically.

FIRST-ADOPTER TEACHERS

The ReMath Project is to be commended for its efforts to study the impact of Casyopée and tasks developed around it on teacher adoption of the tool. Lagrange points out the pressing need to study the dissemination of technological tools through the stages of implementation, including:

- Development with first-adaptor teachers and experts,
- Proclamation by teacher educators as an innovative tool through preservice teacher courses or professional development for teachers in the field,
- Innovation in the classroom by expert teachers,
- Classroom implementation by a community of practitioners (mid-adaptors), and
- Use of the tool by most teachers.

I am always dismayed at the apprehension of preservice teachers and the dearth of implementation by classroom teachers concerning the use of CAS and IG. CAS, IG, and data analysis tools (such as spreadsheets and Fathom) are tools with great potential for improving student understanding, but in the U.S. they are greatly underutilized. Although the implementation barriers identified by Lagrange's – close to curriculum applications, time to implement technology, curriculum coverage issues, school organization issues, and professional development needs – are real problems, I believe that, until we move mathematics instruction from teaching for mastery of isolated skills and procedures to teaching for understanding, improving dissemination of technology tools will be very difficult. Mid-adopters and third-layer teachers, using Lagrange's terms, are pressured by high stakes testing and see these tools as add-ons or potential distractions.

The theoretical constructs addressing the dissemination issue are well taken. The boundary objects construct promotes mutual negotiation and meaning-construction as a norm, and should be a basis of good professional development. This leads to communal design of artifacts by researchers and teachers, as was done in the ReMath project. However, communal design is restrictive, impacting primarily the first-adopter teacher. While I agree that we must develop an understanding of how to engender the process of instrumental genesis, I maintain that to impact mid-adopter and third-layer teachers in the United States we must fundamentally change the expectations for what it means to learn mathematics.

Lagrange's two first-adopters had ownership in the project's work, so they were willing to overlook Casyopée's limitations. They had an intimate understanding of the intent and purpose of the tool, so they could more easily guide their students to desired outcomes while avoiding pitfalls. I venture that the first-adopters possessed an indepth understanding of geometric co-variance and function as model, either before the software development process or because of it. This reduced the complexity of the task for them, allowing them to focus on implementation of the tool in the classroom. For teachers to be successful implementers, they must have an in-depth understanding of the mathematical concept that is the focus of the task as well as experience with how the tool elicits this understanding from students.

DISSEMINATION CHALLENGES

As Lagrange noted, the six mid-adopter teachers had both experience with teaching with technology in the classroom and a positive attitude about the role of technology in teaching mathematics. They did not possess the first-adopters' knowledge of the development of Casyopée. The mid-adopters took six months to learn about the software before being deemed comfortable to implement the lesson. So they had an extended exposure to the tool. They also received mentoring from the expert teachers.

Despite this support, the mid-adopter teachers still displayed some traits of novice users. It appears they were more focused on the procedural and algorithmic aspects of the program then they were on the conceptual development aspects. When they implemented the tool in the classroom they focused on the CAS component and its ability to perform algorithms, ignoring the scenarios on modeling geometric dependency. The mid-adopters failed to comprehend, or at least target, the enduring concept that was the focus of developing the tool.

TWO ADDITIONAL ISSUES

The key question that arises for me from Lagrange’s article is how the mathematics education community can move CAS and IG from use by a relatively small number of innovative first-adopter teachers to tools that are widely used. I believe that the primary barriers to further implementation of CAS and IG tools in mathematics classroom are institutional policy of teaching for knowledge of mathematical algorithms and procedures rather than teaching for mathematical understanding and problem solving, and ineffective professional development.

Until school administrators, teachers, parents, and students accept that teaching for understanding is the central tenant of quality mathematics education, then the application of CAS and IG in the classroom will be seen as extraneous. Current national and international high stakes testing with closed-form items, such as TIMSS, PISA (and NAEP, ACT and SAT in the U.S.), exacerbate the teaching for knowledge dilemma by assessing primarily knowledge and not understanding. Wiggins and McTighe [5] created a backward design process with the goal of engaging students in inquiry and uncovering big ideas of content. If the mathematics education community can help teachers to value understanding developed through student inquiry into big ideas, then there will be a natural role for CAS and IG tools. I submit that the first-adopter teachers in Lagrange’s project already valued inquiry and understanding, so they saw the importance of Casyopée. Moving mid-adopters and third-layer teachers to a pedagogical view of teaching for understanding is an essential criterion for improved dissemination.

Sustained content-based Professional Development (PD) is the second key to dissemination. The National Council of Supervisors of Mathematics [NCSM, 6] produced a summary of effective PD principles that call for school administrators, teachers, and PD providers to collaborate across three interconnected program components:

- **Diagnosing** the instructional problem to be solved. The instructional problem that supports the implementation of CAS and IG is moving from teaching for knowledge to teaching for understanding, which includes actively engaging students in open-ended explorations around enduring mathematical concepts, making CAS and IG tools an essential part of the curriculum, not an add-on.
- **Designing and Implementing** intervention strategies that are appropriate, including how they will be introduced. Research indicates that effects on teachers' classroom practice first appear after 30 clock hours of PD and increase through at least 80 hours. If Casyopée is to be integrated into the classroom, all participating teachers must receive extensive and sustained PD focused on mathematical concepts. The ReMath Project took strong actions in this area and still had teachers struggle to implement the tool.
- **Evaluation** as ascertaining what is working and looking back at what worked. Evaluation should take place on a two-tiered level, change in teacher practice and improvement in student learning. Pre-post data on student understanding of geometric co-variation and how Casyopée impacted conceptual development should be gathered. The peer mentoring aspect implemented by ReMath is recommended by NCSM, including observing lessons taught using Casyopée and experts modeling its use in the peer teacher's classroom.

CLOSING REMARKS

As Lagrange notes, encouraging second adopters and third-layer teachers is non-trivial work. Perhaps providing sustained effective PD and developing a focus on teaching for understanding for inquiry into big mathematical ideas, along with improved technology tools and tasks, will provide a mechanism for improved and extensive classroom implementation of CAS and IG.

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